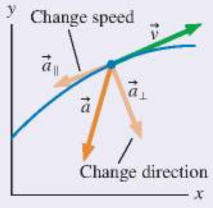
4 Kinematics in Two Dimensions

The water droplets are following the parabolic trajectories of projectile motion.

IN THIS CHAPTER, you will learn how to solve problems about motion in a plane.

How do objects accelerate in two dimensions?

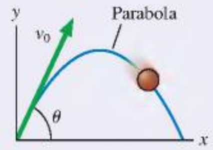
An object accelerates when it changes velocity. In two dimensions, velocity can change by changing magnitude (speed) or by changing direction. These are represented by acceleration components tangent to and perpendicular to an object's trajectory.



« LOOKING BACK Section 1.5 Finding acceleration vectors on a motion diagram

What is projectile motion?

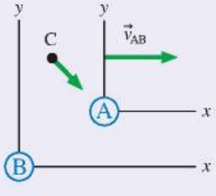
Projectile motion is two-dimensional free-fall motion under the influence of only gravity. Projectile motion follows a parabolic trajectory. It has uniform motion in the horizontal direction and $a_y = -g$ in the vertical direction.



« LOOKING BACK Section 2.5 Free fall

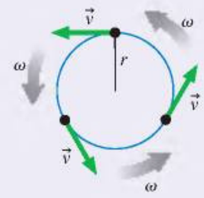
What is relative motion?

Coordinate systems that move relative to each other are called reference frames. If object C has velocity $\vec{v}_{CA}$ relative to a reference frame A, and if A moves with velocity $\vec{v}_{AB}$ relative to another reference frame B, then the velocity of C in reference frame B is $\vec{v}_{CB} = \vec{v}_{CA} + \vec{v}_{AB}$.



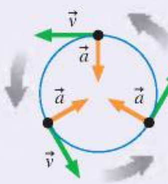
What is circular motion?

An object moving in a circle (or rotating) has an *angular displacement* instead of a linear displacement. Circular motion is described by angular velocity ω (analogous to velocity v_s) and angular acceleration α (analogous to acceleration a_s). We'll study both uniform and accelerated circular motion.



What is centripetal acceleration?

An object in circular motion is always changing direction. The acceleration of changing direction—called centripetal acceleration—points to the center of the circle. All circular motion has a centripetal acceleration. An object also has a *tangential acceleration* if it is changing speed.



Where is two-dimensional motion used?

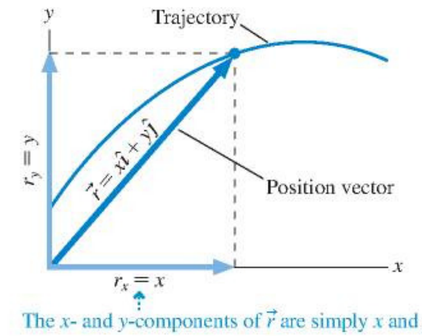
Linear motion allowed us to introduce the concepts of motion, but most real motion takes place in two or even three dimensions. Balls move along curved trajectories, cars turn corners, planets orbit the sun, and electrons spiral in the earth's magnetic field. Where is two-dimensional motion used? Everywhere!

4.1 Motion in Two Dimensions

Motion diagrams are an important tool for visualizing motion, and we'll continue to use them, but we also need to develop a mathematical description of motion in two dimensions. For convenience, we'll say that any two-dimensional motion is in the xy-plane regardless of whether the plane of motion is horizontal or vertical.

FIGURE 4.1 shows a particle moving along a curved path—its *trajectory*—in the xy-plane. We can locate the particle in terms of its position vector $\vec{r} = x\hat{\imath} + y\hat{\jmath}$.

FIGURE 4.1 A particle moving along a trajectory in the xy-plane.



NOTE In Chapter 2 we made extensive use of position-versus-time graphs, either x versus t or y versus t. Figure 4.1, like many of the graphs we'll use in this chapter, is a graph of y versus x. In other words, it's an actual *picture* of the trajectory, not an abstract representation of the motion.

FIGURE 4.2a shows the particle moving from position $\vec{r}_1$ at time t_1 to position $\vec{r}_2$ at a later time t_2. The average velocity—pointing in the direction of the displacement $\Delta\vec{r}$—is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} \tag{4.1}$$

You learned in Chapter 2 that the instantaneous velocity is the limit of $\vec{v}_{avg}$ as $\Delta t \to 0$. As Δt decreases, point 2 moves closer to point 1 until, as **FIGURE 4.2b** shows, the displacement vector becomes tangent to the curve. Consequently, **the instantaneous velocity vector $\vec{v}$ is tangent to the trajectory**.

Mathematically, the limit of Equation 4.1 gives

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} \tag{4.2}$$

We can also write the velocity vector in terms of its x- and y-components as

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \tag{4.3}$$

Comparing Equations 4.2 and 4.3, you can see that the velocity vector $\vec{v}$ has x- and y-components

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt} \tag{4.4}$$

That is, the x-component v_x of the velocity vector is the rate dx/dt at which the particle's x-coordinate is changing. The y-component is similar.

FIGURE 4.2c illustrates another important feature of the velocity vector. If the vector's angle θ is measured from the positive x-direction, the velocity vector components are

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \end{aligned} \tag{4.5}$$

where

$$v = \sqrt{v_x^2 + v_y^2} \tag{4.6}$$

is the particle's *speed* at that point. Speed is always a positive number (or zero), whereas the components are *signed* quantities (i.e., they can be positive or negative) to convey information about the direction of the velocity vector. Conversely, we can use the two velocity components to determine the direction of motion:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tag{4.7}$$

NOTE In Chapter 2, you learned that the *value* of the velocity is the *slope* of the position-versus-time graph. Now we see that the *direction* of the velocity vector $\vec{v}$ is the *tangent* to the y-versus-x graph of the trajectory. **FIGURE 4.3**, on the next page, reminds you that these two graphs use different interpretations of the tangent lines. The tangent to the trajectory does not tell us anything about how fast the particle is moving.

FIGURE 4.2 The instantaneous velocity vector is tangent to the trajectory.

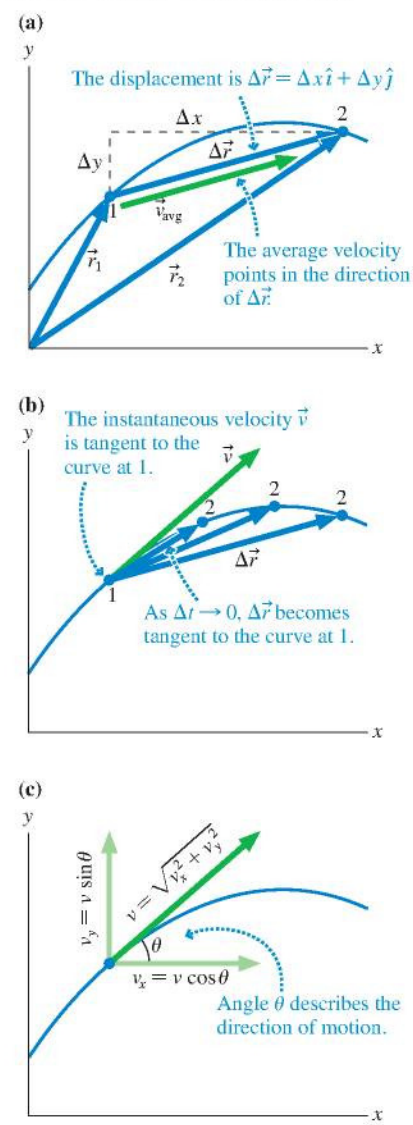
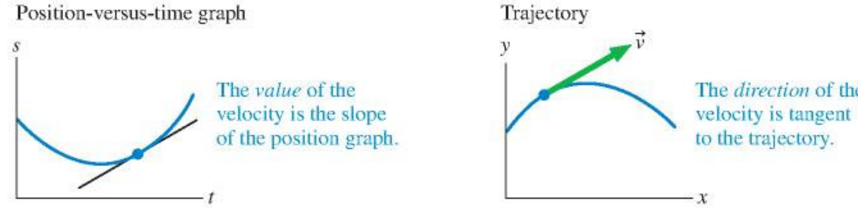


FIGURE 4.3 Two different uses of tangent lines.



EXAMPLE 4.1 **Finding velocity**

A sports car's position on a winding road is given by

$$\vec{r} = (6.0t - 0.10t^2 + 0.00048t^3)\hat{\imath} + (8.0t + 0.060t^2 - 0.00095t^3)\hat{\jmath}$$

where the y-axis points north, t is in s, and r is in m. What are the car's speed and direction at $t = 120$ s?

MODEL Model the car as a particle.

SOLVE Velocity is the derivative of position, so

$$v_x = \frac{dx}{dt} = 6.0 - 2(0.10t) + 3(0.00048t^2)$$

$$v_y = \frac{dy}{dt} = 8.0 + 2(0.060t) - 3(0.00095t^2)$$

Written as a vector, the velocity is

$$\vec{v} = (6.0 - 0.20t + 0.00144t^2)\hat{\imath} + (8.0 + 0.120t - 0.00285t^2)\hat{\jmath}$$

where t is in s and v is in m/s. At $t = 120$ s, we can calculate $\vec{v} = (2.7\hat{\imath} - 18.6\hat{\jmath})$ m/s. The car's speed at this instant is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2.7 \text{ m/s})^2 + (-18.6 \text{ m/s})^2} = 19 \text{ m/s}$$

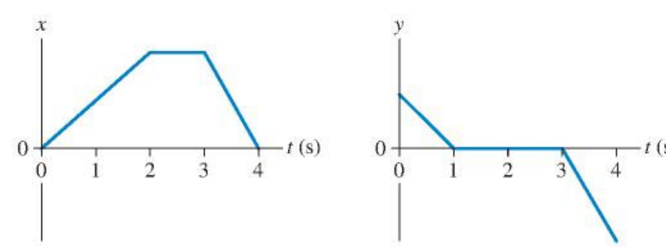
The velocity vector has a negative y-component, so the direction of motion is to the right (east) and down (south). The angle below the x-axis is

$$\theta = \tan^{-1}\left(\frac{|-18.6 \text{ m/s}|}{2.7 \text{ m/s}}\right) = 82°$$

So, at this instant, the car is headed 82° south of east at a speed of 19 m/s.

STOP TO THINK 4.1 During which time interval or intervals is the particle described by these position graphs at rest? More than one may be correct.

a. 0–1 s
b. 1–2 s
c. 2–3 s
d. 3–4 s



Acceleration Graphically

In « Section 1.5 we defined the *average acceleration* $\vec{a}_{avg}$ of a moving object to be

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} \tag{4.8}$$

From its definition, we see that **$\vec{a}$ points in the same direction as $\Delta\vec{v}$,** the change of velocity. As an object moves, its velocity vector can change in two possible ways:

1. The magnitude of $\vec{v}$ can change, indicating a change in speed, or
2. The direction of $\vec{v}$ can change, indicating that the object has changed direction.

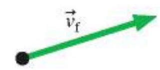
The kinematics of Chapter 2 considered only the acceleration due to changing speed. Now it's time to look at the acceleration associated with changing direction. Tactics Box 4.1 shows how we can use the velocity vectors on a motion diagram to determine the direction of the average acceleration vector. This is an extension of Tactics Box 1.3, which showed how to find $\vec{a}$ for one-dimensional motion.

TACTICS BOX 4.1

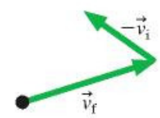
Finding the acceleration vector

To find the acceleration between velocity $\vec{v}_i$ and velocity $\vec{v}_f$:

❶ Draw the velocity vector $\vec{v}_f$.



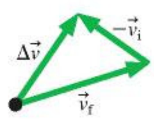
❷ Draw $-\vec{v}_i$ at the tip of $\vec{v}_f$.



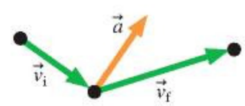
❸ Draw $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$= \vec{v}_f + (-\vec{v}_i)$

This is the direction of $\vec{a}$.



❹ Return to the original motion diagram. Draw a vector at the middle point in the direction of $\Delta\vec{v}$; label it $\vec{a}$. This is the average acceleration between $\vec{v}_i$ and $\vec{v}_f$.



Exercises 1–4

Our everyday use of the word "accelerate" means "speed up." The mathematical definition of acceleration—the rate of change of velocity—also includes slowing down, as you learned in Chapter 2, as well as changing direction. All these are motions that change the velocity.

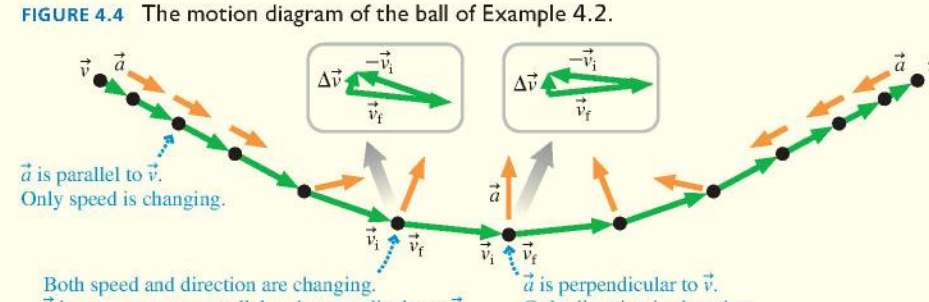
EXAMPLE 4.2 | **Through the valley**

A ball rolls down a long hill, through the valley, and back up the other side. Draw a complete motion diagram of the ball.

MODEL Model the ball as a particle.

VISUALIZE FIGURE 4.4 is the motion diagram. Where the particle moves along a *straight line,* it speeds up if $\vec{a}$ and $\vec{v}$ point in the same direction and slows down if $\vec{a}$ and $\vec{v}$ point in opposite directions. This important idea was the basis for the one-dimensional kinematics we developed in Chapter 2. When the direction of $\vec{v}$ changes, as it does when the ball goes through the valley, we need to use vector subtraction to find the direction of $\Delta\vec{v}$ and thus of $\vec{a}$. The procedure is shown at two points in the motion diagram.

FIGURE 4.4 The motion diagram of the ball of Example 4.2.



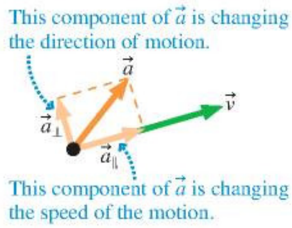
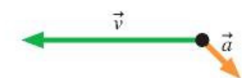


FIGURE 4.5 Analyzing the acceleration vector.

FIGURE 4.5 shows that an object's acceleration vector can be decomposed into a component parallel to the velocity—that is, parallel to the direction of motion—and a component perpendicular to the velocity. **$\vec{a}_\parallel$ is the piece of the acceleration that causes the object to change speed,** speeding up if $\vec{a}_\parallel$ points in the same direction as $\vec{v}$, slowing down if they point in opposite directions. **$\vec{a}_\perp$ is the piece of the acceleration that causes the object to change direction.** An object changing direction *always* has a component of acceleration perpendicular to the direction of motion.

Looking back at Example 4.2, we see that $\vec{a}$ is parallel to $\vec{v}$ on the straight portions of the hill where only speed is changing. At the very bottom, where the ball's direction is changing but not its speed, $\vec{a}$ is perpendicular to $\vec{v}$. The acceleration is angled with respect to velocity—having both parallel and perpendicular components—at those points where both speed and direction are changing.

STOP TO THINK 4.2 This acceleration will cause the particle to



a. Speed up and curve upward.
b. Speed up and curve downward.
c. Slow down and curve upward.
d. Slow down and curve downward.
e. Move to the right and down.
f. Reverse direction.

Acceleration Mathematically

In Tactics Box 4.1, the average acceleration is found from two velocity vectors separated by the time interval Δt. If we let Δt get smaller and smaller, the two velocity vectors get closer and closer. In the limit $\Delta t \rightarrow 0$, we have the instantaneous acceleration $\vec{a}$ at the same point on the trajectory (and the same instant of time) as the instantaneous velocity $\vec{v}$. This is shown in FIGURE 4.6.

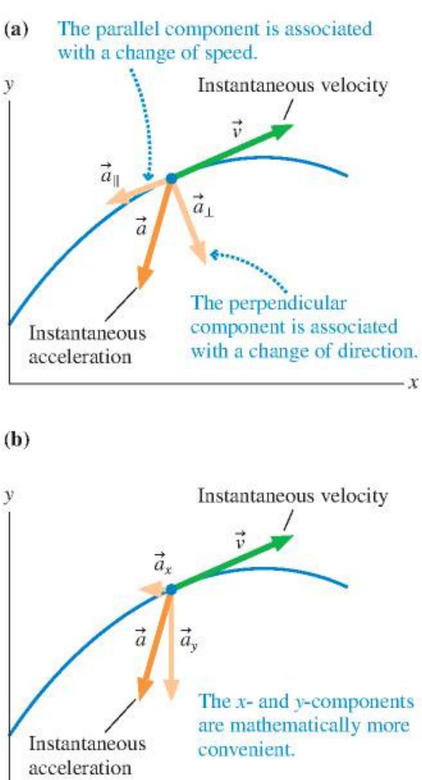


FIGURE 4.6 The instantaneous acceleration $\vec{a}$.

By definition, the acceleration vector $\vec{a}$ is the rate at which the velocity $\vec{v}$ is changing at that instant. To show this, Figure 4.6a decomposes $\vec{a}$ into components $\vec{a}_\parallel$ and $\vec{a}_\perp$ that are parallel and perpendicular to the trajectory. As we just showed, $\vec{a}_\parallel$ is associated with a change of speed, and $\vec{a}_\perp$ is associated with a change of direction. Both kinds of changes are accelerations. Notice that $\vec{a}_\perp$ always points toward the "inside" of the curve because that is the direction in which $\vec{v}$ is changing.

Although the parallel and perpendicular components of $\vec{a}$ convey important ideas about acceleration, it's often more practical to write $\vec{a}$ in terms of the x- and y-components shown in Figure 4.6b. Because $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$, we find

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} \tag{4.9}$$

from which we see that

$$a_x = \frac{dv_x}{dt} \quad \text{and} \quad a_y = \frac{dv_y}{dt} \tag{4.10}$$

That is, the x-component of $\vec{a}$ is the rate dv_x/dt at which the x-component of velocity is changing.

Notice that Figures 4.6a and 4.6b show the *same* acceleration vector; all that differs is how we've chosen to decompose it. For motion with constant acceleration, which includes projectile motion, the decomposition into x- and y-components is most convenient. But we'll find that the parallel and perpendicular components are especially suited to an analysis of circular motion.

Constant Acceleration

If the acceleration $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ is constant, then the two components a_x and a_y are both constant. In this case, everything you learned about constant-acceleration kinematics in « Section 2.4 carries over to two-dimensional motion.

Consider a particle that moves with constant acceleration from an initial position $\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath}$, starting with initial velocity $\vec{v}_i = v_{ix}\hat{\imath} + v_{iy}\hat{\jmath}$. Its position and velocity at a final point f are

$$\begin{aligned} x_f &= x_i + v_{ix}\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2 & \qquad y_f &= y_i + v_{iy}\,\Delta t + \tfrac{1}{2}a_y(\Delta t)^2 \\ v_{fx} &= v_{ix} + a_x\,\Delta t & \qquad v_{fy} &= v_{iy} + a_y\,\Delta t \end{aligned} \qquad (4.11)$$

There are *many* quantities to keep track of in two-dimensional kinematics, making the pictorial representation all the more important as a problem-solving tool.

NOTE For constant acceleration, the x-component of the motion and the y-component of the motion are independent of each other. However, they remain connected through the fact that Δt must be the same for both.

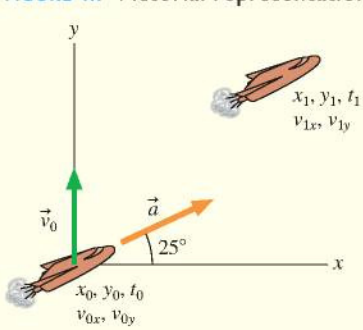
EXAMPLE 4.3 | Plotting a spacecraft trajectory

In the distant future, a small spacecraft is drifting "north" through the galaxy at 680 m/s when it receives a command to return to the starship. The pilot rotates the spacecraft until the nose is pointed 25° north of east, then engages the ion engine. The spacecraft accelerates at 75 m/s^2. Plot the spacecraft's trajectory for the first 20 s.

MODEL Model the spacecraft as a particle with constant acceleration.

VISUALIZE FIGURE 4.7 shows a pictorial representation in which the y-axis points north and the spacecraft starts at the origin. Notice that each point in the motion is labeled with *two* positions (x and y), *two* velocity components (v_x and v_y), and the time t. This will be our standard labeling scheme for trajectory problems.

FIGURE 4.7 Pictorial representation of the spacecraft.



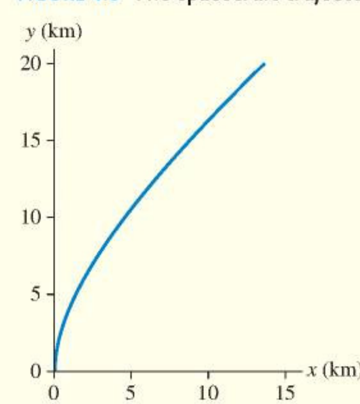
Known
$x_0 = y_0 = 0$ m $\quad v_{0x} = 0$ m/s $\quad v_{0y} = 680$ m/s
$a_x = (75\text{ m/s}^2)\cos 25° = 68.0\text{ m/s}^2$
$a_y = (75\text{ m/s}^2)\sin 25° = 31.6\text{ m/s}^2$
$t_0 = 0$ s $\quad t_1 = 0$ s to 20 s
Find
x_1 and y_1

SOLVE The acceleration vector has both x- and y-components; their values have been calculated in the pictorial representation. But it is a *constant* acceleration, so we can write

$$\begin{aligned} x_1 &= x_0 + v_{0x}(t_1 - t_0) + \tfrac{1}{2}a_x(t_1 - t_0)^2 \\ &= 34.0t_1^2 \text{ m} \\ y_1 &= y_0 + v_{0y}(t_1 - t_0) + \tfrac{1}{2}a_y(t_1 - t_0)^2 \\ &= 680t_1 + 15.8t_1^2 \text{ m} \end{aligned}$$

where t_1 is in s. Graphing software produces the trajectory shown in FIGURE 4.8. The trajectory is a parabola, which is characteristic of two-dimensional motion with constant acceleration.

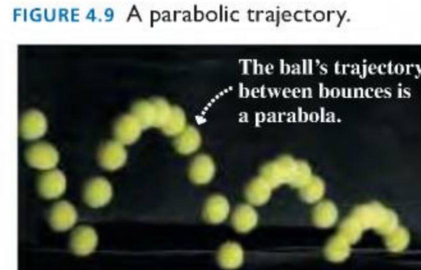
FIGURE 4.8 The spacecraft trajectory.



4.2 Projectile Motion

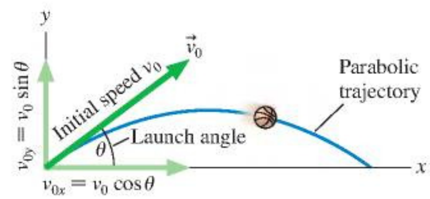
Baseballs and tennis balls flying through the air, Olympic divers, and daredevils shot from cannons all exhibit what we call *projectile motion*. A **projectile is an object that moves in two dimensions under the influence of only gravity.** Projectile motion is an extension of the free-fall motion we studied in Chapter 2. We will continue to neglect the influence of air resistance, leading to results that are a good approximation of reality for relatively heavy objects moving relatively slowly over relatively short distances. As we'll see, projectiles in two dimensions follow a *parabolic trajectory* like the one seen in FIGURE 4.9.

FIGURE 4.9 A parabolic trajectory.



The start of a projectile's motion, be it thrown by hand or shot from a gun, is called the *launch*, and the angle θ of the initial velocity $\vec{v}_0$ above the horizontal (i.e., above

FIGURE 4.10 A projectile launched with initial velocity $\vec{v}_0$.



the x-axis) is called the **launch angle.** FIGURE 4.10 illustrates the relationship between the initial velocity vector $\vec{v}_0$ and the initial values of the components v_{0x} and v_{0y}. You can see that

$$v_{0x} = v_0 \cos\theta$$
$$v_{0y} = v_0 \sin\theta \qquad (4.12)$$

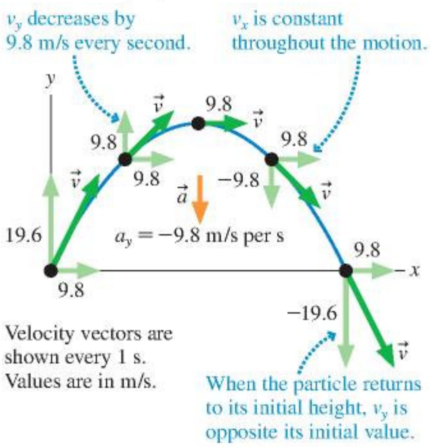
where v_0 is the initial speed.

NOTE A projectile launched at an angle *below* the horizontal (such as a ball thrown downward from the roof of a building) has *negative* values for θ and v_{0y}. However, the *speed* v_0 is always positive.

Gravity acts downward, and we know that objects released from rest fall straight down, not sideways. Hence a projectile has no horizontal acceleration, while its vertical acceleration is simply that of free fall. Thus

$$a_x = 0$$
$$a_y = -g \qquad \text{(projectile motion)} \qquad (4.13)$$

FIGURE 4.11 The velocity and acceleration vectors of a projectile.



In other words, **the vertical component of acceleration a_y is just the familiar $-g$ of free fall, while the horizontal component a_x is zero. Projectiles are in free fall.**

To see how these conditions influence the motion, FIGURE 4.11 shows a projectile launched from $(x_0, y_0) = (0\text{ m}, 0\text{ m})$ with an initial velocity $\vec{v}_0 = (9.8\hat{\imath} + 19.6\hat{\jmath})$ m/s. The value of v_x never changes because there's no horizontal acceleration, but v_y decreases by 9.8 m/s every second. This is what it *means* to accelerate at $a_y = -9.8\text{ m/s}^2 = (-9.8\text{ m/s})$ per second.

You can see from Figure 4.11 that **projectile motion is made up of two independent motions:** uniform motion at constant velocity in the horizontal direction and free-fall motion in the vertical direction. The kinematic equations that describe these two motions are simply Equations 4.11 with $a_x = 0$ and $a_y = -g$.

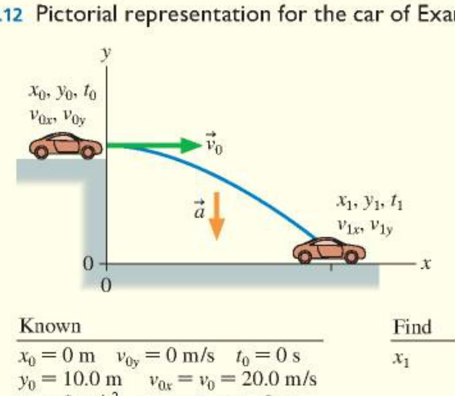
EXAMPLE 4.4 Don't try this at home!

A stunt man drives a car off a 10.0-m-high cliff at a speed of 20.0 m/s. How far does the car land from the base of the cliff?

MODEL Model the car as a particle in free fall. Assume that the car is moving horizontally as it leaves the cliff.

VISUALIZE The pictorial representation, shown in FIGURE 4.12, is *very* important because the number of quantities to keep track of is quite large. We have chosen to put the origin at the base of the cliff. The assumption that the car is moving horizontally as it leaves the cliff leads to $v_{0x} = v_0$ and $v_{0y} = 0$ m/s.

FIGURE 4.12 Pictorial representation for the car of Example 4.4.



SOLVE Each point on the trajectory has x- and y-components of position, velocity, and acceleration but only *one* value of time. The time needed to move horizontally to x_1 is the *same* time needed to fall vertically through distance y_0. **Although the horizontal and vertical motions are independent, they are connected through the time t.** This is a critical observation for solving projectile motion problems. The kinematics equations with $a_x = 0$ and $a_y = -g$ are

$$x_1 = x_0 + v_{0x}(t_1 - t_0) = v_0 t_1$$
$$y_1 = 0 = y_0 + v_{0y}(t_1 - t_0) - \tfrac{1}{2}g(t_1 - t_0)^2 = y_0 - \tfrac{1}{2}g t_1^{\,2}$$

We can use the vertical equation to determine the time t_1 needed to fall distance y_0:

$$t_1 = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(10.0\text{ m})}{9.80\text{ m/s}^2}} = 1.43\text{ s}$$

We then insert this expression for t into the horizontal equation to find the distance traveled:

$$x_1 = v_0 t_1 = (20.0\text{ m/s})(1.43\text{ s}) = 28.6\text{ m}$$

ASSESS The cliff height is ≈ 33 ft and the initial speed is $v_0 \approx 40$ mph. Traveling $x_1 = 29\text{ m} \approx 95$ ft before hitting the ground seems reasonable.

The x- and y-equations of Example 4.4 are parametric equations. It's not hard to eliminate t and write an expression for y as a function of x. From the x_1 equation, $t_1 = x_1/v_0$. Substituting this into the y_1 equation, we find

$$y = y_0 - \frac{g}{2v_0^2}x^2 \tag{4.14}$$

The graph of $y = cx^2$ is a parabola, so Equation 4.14 represents an inverted parabola that starts from height y_0. This proves, as we asserted previously, that a projectile follows a parabolic trajectory.

Reasoning About Projectile Motion

Suppose a heavy ball is launched exactly horizontally at height h above a horizontal field. At the exact instant that the ball is launched, a second ball is simply dropped from height h. Which ball hits the ground first?

It may seem hard to believe, but—if air resistance is neglected—the balls hit the ground *simultaneously*. They do so because the horizontal and vertical components of projectile motion are independent of each other. The initial horizontal velocity of the first ball has *no* influence over its vertical motion. Neither ball has any initial motion in the vertical direction, so both fall distance h in the same amount of time. You can see this in **FIGURE 4.13**.

FIGURE 4.13 A projectile launched horizontally falls in the same time as a projectile that is released from rest.

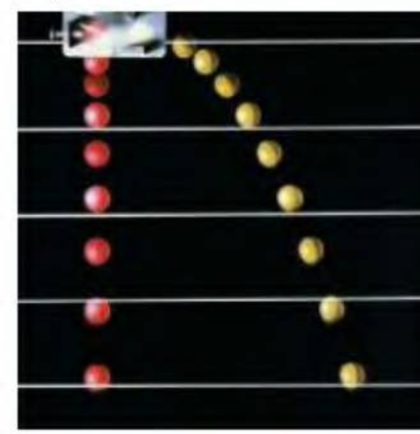


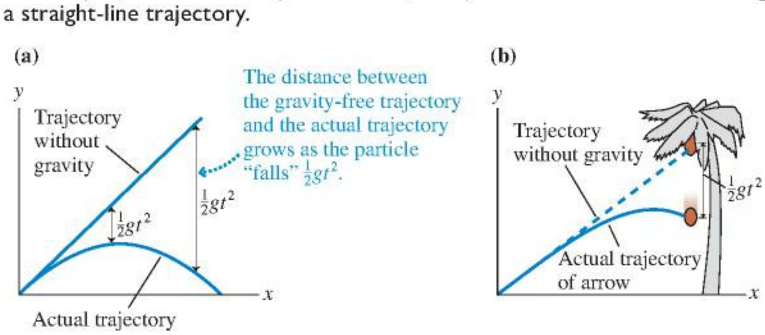
FIGURE 4.14a shows a useful way to think about the trajectory of a projectile. Without gravity, a projectile would follow a straight line. Because of gravity, the particle at time t has "fallen" a distance $\frac{1}{2}gt^2$ below this line. The separation grows as $\frac{1}{2}gt^2$, giving the trajectory its parabolic shape.

Use this idea to think about the following "classic" problem in physics:

> A hungry bow-and-arrow hunter in the jungle wants to shoot down a coconut that is hanging from the branch of a tree. He points his arrow directly at the coconut, but as luck would have it, the coconut falls from the branch at the *exact* instant the hunter releases the string. Does the arrow hit the coconut?

You might think that the arrow will miss the falling coconut, but it doesn't. Although the arrow travels very fast, it follows a slightly curved parabolic trajectory, not a straight line. Had the coconut stayed on the tree, the arrow would have curved under its target as gravity caused it to fall a distance $\frac{1}{2}gt^2$ below the straight line. But $\frac{1}{2}gt^2$ is also the distance the coconut falls while the arrow is in flight. Thus, as **FIGURE 4.14b** shows, the arrow and the coconut fall the same distance and meet at the same point!

FIGURE 4.14 A projectile follows a parabolic trajectory because it "falls" a distance $\frac{1}{2}gt^2$ below a straight-line trajectory.



The Projectile Motion Model

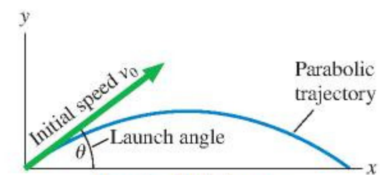
Projectile motion is an ideal that's rarely achieved by real objects. Nonetheless, the **projectile motion model** is another important simplification of reality that we can add to our growing list of models.

MODEL 4.1

Projectile motion

For motion under the influence of only gravity.

- Model the object as a particle launched with speed v_0 at angle θ:
- Mathematically:
  - **Uniform motion** in the horizontal direction with $v_x = v_0 \cos\theta$.
  - **Constant acceleration** in the vertical direction with $a_y = -g$.
  - Same Δt for both motions.



A projectile follows a parabolic trajectory.

- Limitations: Model fails if air resistance is significant.

Exercise 9

PROBLEM-SOLVING STRATEGY 4.1

Projectile motion problems

MODEL Is it reasonable to ignore air resistance? If so, use the projectile motion model.

VISUALIZE Establish a coordinate system with the x-axis horizontal and the y-axis vertical. Define symbols and identify what the problem is trying to find. For a launch at angle θ, the initial velocity components are $v_{ix} = v_0 \cos\theta$ and $v_{iy} = v_0 \sin\theta$.

SOLVE The acceleration is known: $a_x = 0$ and $a_y = -g$. Thus the problem is one of two-dimensional kinematics. The kinematic equations are

Horizontal	Vertical
$x_f = x_i + v_{ix}\,\Delta t$	$y_f = y_i + v_{iy}\,\Delta t - \frac{1}{2}g(\Delta t)^2$
$v_{fx} = v_{ix} = \text{constant}$	$v_{fy} = v_{iy} - g\,\Delta t$

Δt is the same for the horizontal and vertical components of the motion. Find Δt from one component, then use that value for the other component.

ASSESS Check that your result has correct units and significant figures, is reasonable, and answers the question.

EXAMPLE 4.5 Jumping frog contest

Frogs, with their long, strong legs, are excellent jumpers. And thanks to the good folks of Calaveras County, California, who have a jumping frog contest every year in honor of a Mark Twain story, we have very good data on how far a determined frog can jump.

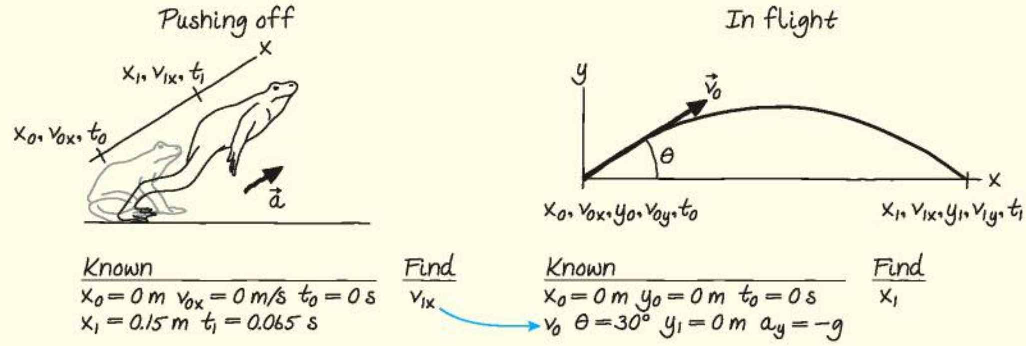
High-speed cameras show that a good jumper goes into a crouch, then rapidly extends his legs by typically 15 cm during a 65 ms push off, leaving the ground at a 30° angle. How far does this frog leap?

MODEL Model the push off as linear motion with uniform acceleration. A bullfrog is fairly heavy and dense, so ignore air resistance and model the leap as projectile motion.

VISUALIZE This is a two-part problem: linear acceleration followed by projectile motion. A key observation is that **the final velocity for pushing off the ground becomes the initial velocity of the projectile motion**. FIGURE 4.15 shows a separate pictorial representation for each part. Notice that we've used different coordinate systems for the two parts; coordinate systems are our choice, and for each part of the motion we've chosen the coordinate system that makes the problem easiest to solve.

SOLVE While pushing off, the frog travels 15 cm = 0.15 m in 65 ms = 0.065 s. We could find his speed at the end of pushing off if we knew the acceleration. Because the initial velocity is zero,

FIGURE 4.15 Pictorial representations of the jumping frog.



we can find the acceleration from the position-acceleration-time kinematic equation:

$$x_1 = x_0 + v_{0x}\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2 = \tfrac{1}{2}a_x(\Delta t)^2$$

$$a_x = \frac{2x_1}{(\Delta t)^2} = \frac{2(0.15\text{ m})}{(0.065\text{ s})^2} = 71\text{ m/s}^2$$

This is a substantial acceleration, but it doesn't last long. At the end of the 65 ms push off, the frog's velocity is

$$v_{1x} = v_{0x} + a_x\,\Delta t = (71\text{ m/s}^2)(0.065\text{ s}) = 4.62\text{ m/s}$$

We'll keep an extra significant figure here to avoid round-off error in the second half of the problem.

The end of the push off is the beginning of the projectile motion, so the second part of the problem is to find the distance of a projectile launched with velocity $\vec{v}_0 = (4.62\text{ m/s}, 30°)$. The initial x- and y-components of the launch velocity are

$$v_{0x} = v_0\cos\theta \qquad v_{0y} = v_0\sin\theta$$

The kinematic equations of projectile motion, with $a_x = 0$ and $a_y = -g$, are

$$x_1 = x_0 + v_{0x}\,\Delta t = (v_0\cos\theta)\Delta t \qquad y_1 = y_0 + v_{0y}\,\Delta t - \tfrac{1}{2}g(\Delta t)^2 = (v_0\sin\theta)\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

We can find the time of flight from the vertical equation by setting $y_1 = 0$:

$$0 = (v_0\sin\theta)\Delta t - \tfrac{1}{2}g(\Delta t)^2 = (v_0\sin\theta - \tfrac{1}{2}g\,\Delta t)\Delta t$$

and thus

$$\Delta t = 0 \quad \text{or} \quad \Delta t = \frac{2v_0\sin\theta}{g}$$

Both are legitimate solutions. The first corresponds to the instant when $y = 0$ at the launch, the second to when $y = 0$ as the frog hits the ground. Clearly, we want the second solution. Substituting this expression for Δt into the equation for x_1 gives

$$x_1 = (v_0\cos\theta)\frac{2v_0\sin\theta}{g} = \frac{2v_0^2\sin\theta\cos\theta}{g}$$

We can simplify this result with the trigonometric identity $2\sin\theta\cos\theta = \sin(2\theta)$. Thus the distance traveled by the frog is

$$x_1 = \frac{v_0^2\sin(2\theta)}{g}$$

Using $v_0 = 4.62$ m/s and $\theta = 30°$, we find that the frog leaps a distance of 1.9 m.

ASSESS 1.9 m is about 6 feet, or about 10 times the frog's body length. That's pretty amazing, but true. Jumps of 2.2 m have been recorded in the lab. And the Calaveras County record holder, Rosie the Ribeter, covered 6.5 m—21 feet—in three jumps!

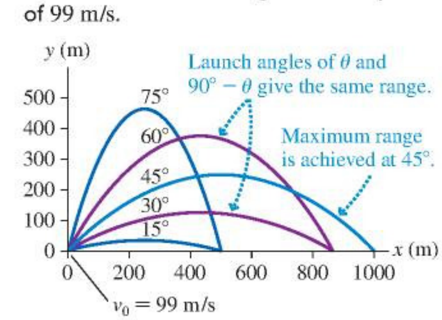
The distance a projectile travels is called its *range*. As Example 4.5 found, a projectile that lands at the same elevation from which it was launched has

$$\text{range} = \frac{v_0^2\sin(2\theta)}{g} \tag{4.15}$$

The maximum range occurs for $\theta = 45°$, where $\sin(2\theta) = 1$. But there's more that we can learn from this equation. Because $\sin(180° - x) = \sin x$, it follows that $\sin(2(90° - \theta)) = \sin(2\theta)$. Consequently, a projectile launched either at angle θ *or* at angle $(90° - \theta)$ will travel the same distance *over level ground*. FIGURE 4.16 shows several trajectories of projectiles launched with the same initial speed.

NOTE Equation 4.15 is *not* a general result. It applies *only* in situations where the projectile lands at the same elevation from which it was fired.

FIGURE 4.16 Trajectories of a projectile launched at different angles with a speed of 99 m/s.



STOP TO THINK 4.3 A 50 g marble rolls off a table and hits 2 m from the base of the table. A 100 g marble rolls off the same table with the same speed. It lands at distance

a. Less than 1 m. b. 1 m. c. Between 1 m and 2 m.
d. 2 m. e. Between 2 m and 4 m. f. 4 m.

4.3 Relative Motion

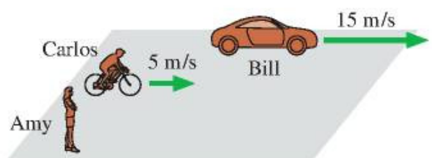


FIGURE 4.17 Velocities in Amy's reference frame.

FIGURE 4.17 shows Amy and Bill watching Carlos on his bicycle. According to Amy, Carlos's velocity is $v_x = 5$ m/s. Bill sees the bicycle receding in his rearview mirror, in the *negative* x-direction, getting 10 m farther away from him every second. According to Bill, Carlos's velocity is $v_x = -10$ m/s. Which is Carlos's *true* velocity?

Velocity is not a concept that can be true or false. Carlos's velocity *relative to Amy* is $(v_x)_{CA} = 5$ m/s, where the subscript notation means "C relative to A." Similarly, Carlos's velocity *relative to Bill* is $(v_x)_{CB} = -10$ m/s. These are both valid descriptions of Carlos's motion.

It's not hard to see how to combine the velocities for one-dimensional motion:

The first subscript is the same on both sides.
The last subscript is the same on both sides.

$$(v_x)_{CB} = (v_x)_{CA} + (v_x)_{AB} \tag{4.16}$$

The inner subscripts "cancel."

We'll justify this relationship later in this section and then extend it to two-dimensional motion.

Equation 4.16 tells us that the velocity of C relative to B is the velocity of C relative to A *plus* the velocity of A relative to B. Note that

$$(v_x)_{AB} = -(v_x)_{BA} \tag{4.17}$$

because if B is moving to the right relative to A, then A is moving to the left relative to B. In Figure 4.17, Bill is moving to the right relative to Amy with $(v_x)_{BA} = 15$ m/s, so $(v_x)_{AB} = -15$ m/s. Knowing that Carlos's velocity relative to Amy is 5 m/s, we find that Carlos's velocity relative to Bill is, as expected, $(v_x)_{CB} = (v_x)_{CA} + (v_x)_{AB} = 5\text{ m/s} + (-15)\text{ m/s} = -10$ m/s.

EXAMPLE 4.6 A speeding bullet

The police are chasing a bank robber. While driving at 50 m/s, they fire a bullet to shoot out a tire of his car. The police gun shoots bullets at 300 m/s. What is the bullet's speed as measured by a TV camera crew parked beside the road?

MODEL Assume that all motion is in the positive x-direction. The bullet is the object that is observed from both the police car and the ground.

SOLVE The bullet B's velocity relative to the gun G is $(v_x)_{BG} = 300$ m/s. The gun, inside the car, is traveling relative to the TV crew C at $(v_x)_{GC} = 50$ m/s. We can combine these values to find that the bullet's velocity relative to the TV crew on the ground is

$$(v_x)_{BC} = (v_x)_{BG} + (v_x)_{GC} = 300\text{ m/s} + 50\text{ m/s} = 350\text{ m/s}$$

ASSESS It should be no surprise in this simple situation that we simply add the velocities.

Reference Frames

A coordinate system in which an experimenter (possibly with the assistance of helpers) makes position and time measurements of physical events is called a **reference frame.** In Figure 4.17, Amy and Bill each had their own reference frame (where they were at rest) in which they measured Carlos's velocity.

More generally, FIGURE 4.18 shows two reference frames, A and B, and an object C. It is assumed that the reference frames are moving with respect to each other. At this instant of time, the position vector of C in reference frame A is $\vec{r}_{CA}$, meaning "the position of C relative to the origin of frame A." Similarly, $\vec{r}_{CB}$ is the position vector of C in reference frame B. Using vector addition, you can see that

$$\vec{r}_{CB} = \vec{r}_{CA} + \vec{r}_{AB} \tag{4.18}$$

where $\vec{r}_{AB}$ locates the origin of A relative to the origin of B.

In general, object C is moving relative to both reference frames. To find its velocity in each reference frame, take the time derivative of Equation 4.18:

$$\frac{d\vec{r}_{CB}}{dt} = \frac{d\vec{r}_{CA}}{dt} + \frac{d\vec{r}_{AB}}{dt} \tag{4.19}$$

By definition, $d\vec{r}/dt$ is a velocity. The first derivative is $\vec{v}_{CB}$, the velocity of C relative to B. Similarly, the second derivative is the velocity of C relative to A, $\vec{v}_{CA}$. The last derivative is slightly different because it doesn't refer to object C. Instead, this is the velocity $\vec{v}_{AB}$ of reference frame A relative to reference frame B. As we noted in one dimension, $\vec{v}_{AB} = -\vec{v}_{BA}$.

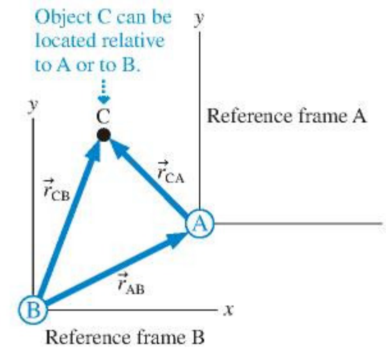
Writing Equation 4.19 in terms of velocities, we have

$$\vec{v}_{CB} = \vec{v}_{CA} + \vec{v}_{AB} \tag{4.20}$$

This relationship between velocities in different reference frames was recognized by Galileo in his pioneering studies of motion, hence it is known as the **Galilean transformation of velocity.** If you know an object's velocity in one reference frame, you can *transform* it into the velocity that would be measured in a different reference frame. Just as in one dimension, the velocity of C relative to B is the velocity of C relative to A plus the velocity of A relative to B, *but* you must add the velocities as vectors for two-dimensional motion.

As we've seen, the Galilean velocity transformation is pretty much common sense for one-dimensional motion. The real usefulness appears when an object travels in a *medium* moving with respect to the earth. For example, a boat moves relative to the water. What is the boat's net motion if the water is a flowing river? Airplanes fly relative to the air, but the air at high altitudes often flows at high speed. Navigation of boats and planes requires knowing both the motion of the vessel in the medium and the motion of the medium relative to the earth.

FIGURE 4.18 Two reference frames.

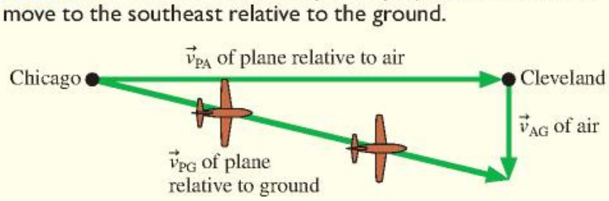


EXAMPLE 4.7 Flying to Cleveland I

Cleveland is 300 miles east of Chicago. A plane leaves Chicago flying due east at 500 mph. The pilot forgot to check the weather and doesn't know that the wind is blowing to the south at 50 mph. What is the plane's ground speed? Where is the plane 0.60 h later, when the pilot expects to land in Cleveland?

MODEL Establish a coordinate system with the x-axis pointing east and the y-axis north. The plane P flies in the air, so its velocity relative to the air A is $\vec{v}_{PA} = 500\hat{\imath}$ mph. Meanwhile, the air is moving relative to the ground G at $\vec{v}_{AG} = -50\hat{\jmath}$ mph.

FIGURE 4.19 The wind causes a plane flying due east in the air to move to the southeast relative to the ground.



SOLVE The velocity equation $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$ is a vector-addition equation. FIGURE 4.19 shows graphically what happens. Although the nose of the plane points east, the wind carries the plane in a direction somewhat south of east. The plane's velocity relative to the ground is

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG} = (500\hat{\imath} - 50\hat{\jmath}) \text{ mph}$$

The plane's ground speed is

$$v = \sqrt{(v_x)_{PG}^{\,2} + (v_y)_{PG}^{\,2}} = 502 \text{ mph}$$

After flying for 0.60 h at this velocity, the plane's location (relative to Chicago) is

$$x = (v_x)_{PG}t = (500 \text{ mph})(0.60 \text{ h}) = 300 \text{ mi}$$
$$y = (v_y)_{PG}t = (-50 \text{ mph})(0.60 \text{ h}) = -30 \text{ mi}$$

The plane is 30 mi due south of Cleveland! Although the pilot thought he was flying to the east, his actual heading has been $\tan^{-1}(50 \text{ mph}/500 \text{ mph}) = \tan^{-1}(0.10) = 5.71°$ south of east.

EXAMPLE 4.8 Flying to Cleveland II

A wiser pilot flying from Chicago to Cleveland on the same day plots a course that will take her directly to Cleveland. In which direction does she fly the plane? How long does it take to reach Cleveland?

MODEL Establish a coordinate system with the x-axis pointing east and the y-axis north. The air is moving relative to the ground at $\vec{v}_{AG} = -50\hat{\jmath}$ mph.

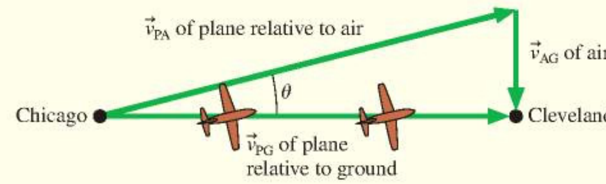
SOLVE The objective of navigation is to move between two points on the earth's surface. The wiser pilot, who knows that the wind will affect her plane, draws the vector picture of FIGURE 4.20. She sees that she'll need $(v_y)_{PG} = 0$, in order to fly due east to Cleveland. This will require turning the nose of the plane at an angle θ north of east, making $\vec{v}_{PA} = (500\cos\theta\hat{\imath} + 500\sin\theta\hat{\jmath})$ mph.

The velocity equation is $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$. The desired heading is found from setting the y-component of this equation to zero:

$$(v_y)_{PG} = (v_y)_{PA} + (v_y)_{AG} = (500\sin\theta - 50)\text{ mph} = 0\text{ mph}$$

$$\theta = \sin^{-1}\left(\frac{50\text{ mph}}{500\text{ mph}}\right) = 5.74°$$

FIGURE 4.20 To travel due east in a south wind, a pilot has to point the plane somewhat to the northeast.



The plane's velocity relative to the ground is then $\vec{v}_{PG} = (500\text{ mph}) \times \cos 5.74°\hat{\imath} = 497\hat{\imath}$ mph. This is slightly slower than the speed relative to the air. The time needed to fly to Cleveland at this speed is

$$t = \frac{300\text{ mi}}{497\text{ mph}} = 0.604\text{ h}$$

It takes 0.004 h = 14 s longer to reach Cleveland than it would on a day without wind.

ASSESS A boat crossing a river or an ocean current faces the same difficulties. These are exactly the kinds of calculations performed by pilots of boats and planes as part of navigation.

STOP TO THINK 4.4 A plane traveling horizontally to the right at 100 m/s flies past a helicopter that is going straight up at 20 m/s. From the helicopter's perspective, the plane's direction and speed are

a. Right and up, less than 100 m/s.
b. Right and up, 100 m/s.
c. Right and up, more than 100 m/s.
d. Right and down, less than 100 m/s.
e. Right and down, 100 m/s.
f. Right and down, more than 100 m/s.

4.4 Uniform Circular Motion

FIGURE 4.21 A particle in uniform circular motion.

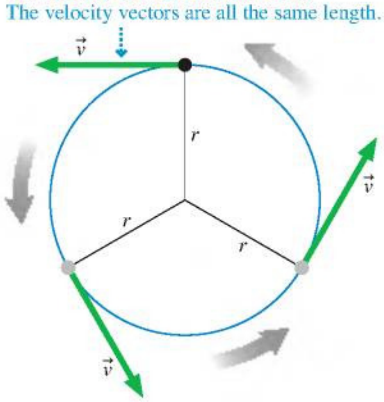


FIGURE 4.21 shows a particle moving around a circle of radius r. The particle might be a satellite in an orbit, a ball on the end of a string, or even just a dot painted on the side of a rotating wheel. Circular motion is another example of motion in a plane, but it is quite different from projectile motion.

To begin the study of circular motion, consider a particle that moves at *constant speed* around a circle of radius r. This is called **uniform circular motion.** Regardless of what the particle represents, its velocity vector $\vec{v}$ is always tangent to the circle. The particle's speed v is constant, so vector $\vec{v}$ is always the same length.

The time interval it takes the particle to go around the circle once, completing one revolution (abbreviated rev), is called the **period** of the motion. Period is represented by the symbol T. It's easy to relate the particle's period T to its speed v. For a particle moving with constant speed, speed is simply distance/time. In one period, the particle moves once around a circle of radius r and travels the circumference $2\pi r$. Thus

$$v = \frac{1\text{ circumference}}{1\text{ period}} = \frac{2\pi r}{T} \tag{4.21}$$

EXAMPLE 4.9 A rotating crankshaft

A 4.0-cm-diameter crankshaft turns at 2400 rpm (revolutions per minute). What is the speed of a point on the surface of the crankshaft?

SOLVE We need to determine the time it takes the crankshaft to make 1 rev. First, we convert 2400 rpm to revolutions per second:

$$\frac{2400 \text{ rev}}{1 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}} = 40 \text{ rev/s}$$

If the crankshaft turns 40 times in 1 s, the time for 1 rev is

$$T = \frac{1}{40} \text{ s} = 0.025 \text{ s}$$

Thus the speed of a point on the surface, where $r = 2.0 \text{ cm} = 0.020 \text{ m}$, is

$$v = \frac{2\pi r}{T} = \frac{2\pi(0.020 \text{ m})}{0.025 \text{ s}} = 5.0 \text{ m/s}$$

Angular Position

Rather than using xy-coordinates, it will be more convenient to describe the position of a particle in circular motion by its distance r from the center of the circle and its angle θ from the positive x-axis. This is shown in **FIGURE 4.22**. The angle θ is the **angular position** of the particle.

We can distinguish a position above the x-axis from a position that is an equal angle below the x-axis by *defining* θ to be positive when measured *counterclockwise* (ccw) from the positive x-axis. An angle measured clockwise (cw) from the positive x-axis has a negative value. "Clockwise" and "counterclockwise" in circular motion are analogous, respectively, to "left of the origin" and "right of the origin" in linear motion, which we associated with negative and positive values of x. A particle 30° below the positive x-axis is equally well described by either $\theta = -30°$ or $\theta = +330°$. We could also describe this particle by $\theta = \frac{11}{12}$ rev, where *revolutions* are another way to measure the angle.

Although degrees and revolutions are widely used measures of angle, mathematicians and scientists usually find it more useful to measure the angle θ in Figure 4.22 by using the **arc length** s that the particle travels along the edge of a circle of radius r. We define the angular unit of **radians** such that

$$\theta(\text{radians}) \equiv \frac{s}{r} \tag{4.22}$$

The radian, which is abbreviated rad, is the SI unit of angle. An angle of 1 rad has an arc length s exactly equal to the radius r.

The arc length completely around a circle is the circle's circumference $2\pi r$. Thus the angle of a full circle is

$$\theta_{\text{full circle}} = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

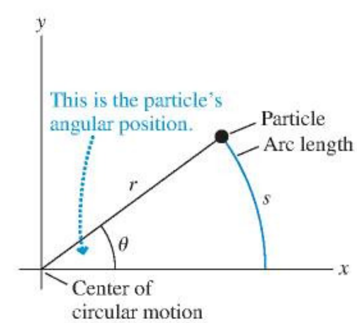
This relationship is the basis for the well-known conversion factors

$$1 \text{ rev} = 360° = 2\pi \text{ rad}$$

As a simple example of converting between radians and degrees, let's convert an angle of 1 rad to degrees:

$$1 \text{ rad} = 1 \text{ rad} \times \frac{360°}{2\pi \text{ rad}} = 57.3°$$

FIGURE 4.22 A particle's position is described by distance r and angle θ.



Circular motion is one of the most common types of motion.

Thus a rough approximation is 1 rad $\approx$ 60°. We will often specify angles in degrees, but keep in mind that the SI unit is the radian.

An important consequence of Equation 4.22 is that the arc length spanning angle θ is

$$s = r\theta \qquad \text{(with } \theta \text{ in rad)} \tag{4.23}$$

This is a result that we will use often, but it is valid *only* if θ is measured in radians and not in degrees. This very simple relationship between angle and arc length is one of the primary motivations for using radians.

NOTE Units of angle are often troublesome. Unlike the kilogram or the second, for which we have standards, the radian is a *defined* unit. It's really just a *name* to remind us that we're dealing with an angle. Consequently, the radian unit sometimes appears or disappears without warning. This seems rather mysterious until you get used to it. This textbook will call your attention to such behavior the first few times it occurs. With a little practice, you'll soon learn when the rad unit is needed and when it's not.

Angular Velocity

FIGURE 4.23 A particle moves with angular velocity ω.

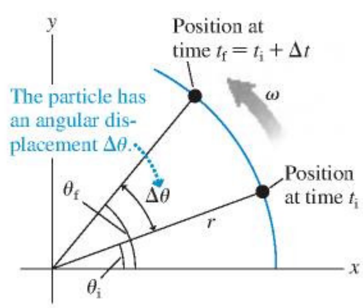


FIGURE 4.23 shows a particle moving in a circle from an initial angular position θ_i at time t_i to a final angular position θ_f at a later time t_f. The change $\Delta\theta = \theta_f - \theta_i$ is called the **angular displacement.** We can measure the particle's circular motion in terms of the rate of change of θ, just as we measured the particle's linear motion in terms of the rate of change of its position s.

In analogy with linear motion, let's define the *average angular velocity* to be

$$\text{average angular velocity} \equiv \frac{\Delta\theta}{\Delta t} \tag{4.24}$$

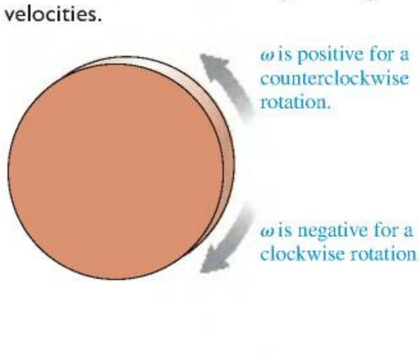
As the time interval Δt becomes very small, $\Delta t \to 0$, we arrive at the definition of the instantaneous **angular velocity:**

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(angular velocity)} \tag{4.25}$$

The symbol ω is a lowercase Greek omega, *not* an ordinary w. The SI unit of angular velocity is rad/s, but °/s, rev/s, and rev/min are also common units. Revolutions per minute is abbreviated rpm.

Angular velocity is the *rate* at which a particle's angular position is changing as it moves around a circle. A particle that starts from $\theta = 0$ rad with an angular velocity of 0.5 rad/s will be at angle $\theta = 0.5$ rad after 1 s, at $\theta = 1.0$ rad after 2 s, at $\theta = 1.5$ rad after 3 s, and so on. Its angular position is increasing at the *rate* of 0.5 radian per second. **A particle moves with uniform circular motion if and only if its angular velocity ω is constant and unchanging.**

FIGURE 4.24 Positive and negative angular velocities.



Angular velocity, like the velocity v_s of one-dimensional motion, can be positive or negative. The signs shown in FIGURE 4.24 are based on the fact that θ was defined to be positive for a counterclockwise rotation. Because the definition $\omega = d\theta/dt$ for circular motion parallels the definition $v_s = ds/dt$ for linear motion, the graphical relationships we found between v_s and s in Chapter 2 apply equally well to ω and θ:

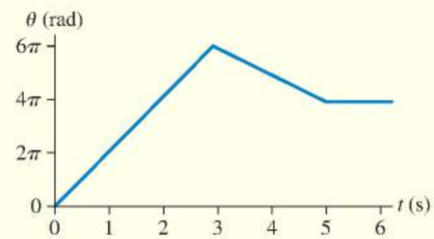
$$\begin{aligned} \omega &= \text{slope of the } \theta\text{-versus-}t \text{ graph at time } t \\ \theta_f &= \theta_i + \text{area under the } \omega\text{-versus-}t \text{ curve between } t_i \text{ and } t_f \\ &= \theta_i + \omega\Delta t \end{aligned} \tag{4.26}$$

You will see many more instances where circular motion is analogous to linear motion with angular variables replacing linear variables. Thus much of what you learned about linear kinematics carries over to circular motion.

EXAMPLE 4.10 A graphical representation of circular motion

FIGURE 4.25 shows the angular position of a painted dot on the edge of a rotating wheel. Describe the wheel's motion and draw an ω-versus-t graph.

FIGURE 4.25 Angular position graph for the wheel of Example 4.10.

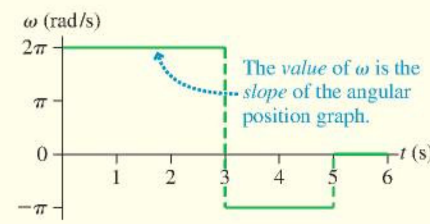


SOLVE Although circular motion seems to "start over" every revolution (every 2π rad), the angular position θ continues to increase. $\theta = 6\pi$ rad corresponds to three revolutions. This wheel makes 3 ccw rev (because θ is getting more positive) in 3 s, immediately reverses direction and makes 1 cw rev in 2 s, then stops at $t = 5$ s and holds the position $\theta = 4\pi$ rad. The angular velocity is found by measuring the slope of the graph:

$$t = 0\text{–}3\text{ s} \qquad \text{slope} = \Delta\theta/\Delta t = 6\pi\text{ rad}/3\text{ s} = 2\pi\text{ rad/s}$$
$$t = 3\text{–}5\text{ s} \qquad \text{slope} = \Delta\theta/\Delta t = -2\pi\text{ rad}/2\text{ s} = -\pi\text{ rad/s}$$
$$t > 5\text{ s} \qquad \text{slope} = \Delta\theta/\Delta t = 0\text{ rad/s}$$

These results are shown as an ω-versus-t graph in FIGURE 4.26. For the first 3 s, the motion is uniform circular motion with $\omega = 2\pi$ rad/s. The wheel then changes to a different uniform circular motion with $\omega = -\pi$ rad/s for 2 s, then stops.

FIGURE 4.26 ω-versus-t graph for the wheel of Example 4.10.



NOTE In physics, we nearly always want to give results as numerical values. Example 4.9 had a π in the equation, but we used its numerical value to compute $v = 5.0$ m/s. However, angles in radians are an exception to this rule. It's okay to leave a π in the value of θ or ω, and we have done so in Example 4.10.

Not surprisingly, the angular velocity ω is closely related to the period and speed of the motion. As a particle goes around a circle one time, its angular displacement is $\Delta\theta = 2\pi$ rad during the interval $\Delta t = T$. Thus, using the definition of angular velocity, we find

$$|\omega| = \frac{2\pi\text{ rad}}{T} \quad \text{or} \quad T = \frac{2\pi\text{ rad}}{|\omega|} \tag{4.27}$$

The period alone gives only the absolute value of $|\omega|$. You need to know the direction of motion to determine the sign of ω.

EXAMPLE 4.11 At the roulette wheel

A small steel roulette ball rolls ccw around the inside of a 30-cm-diameter roulette wheel. The ball completes 2.0 rev in 1.20 s.

a. What is the ball's angular velocity?

b. What is the ball's position at $t = 2.0$ s? Assume $\theta_i = 0$.

MODEL Model the ball as a particle in uniform circular motion.

SOLVE a. The period of the ball's motion, the time for 1 rev, is $T = 0.60$ s. Angular velocity is positive for ccw motion, so

$$\omega = \frac{2\pi\text{ rad}}{T} = \frac{2\pi\text{ rad}}{0.60\text{ s}} = 10.47\text{ rad/s}$$

b. The ball starts at $\theta_i = 0$ rad. After $\Delta t = 2.0$ s, its position is

$$\theta_f = 0\text{ rad} + (10.47\text{ rad/s})(2.0\text{ s}) = 20.94\text{ rad}$$

where we've kept an extra significant figure to avoid round-off error. Although this is a mathematically acceptable answer, an observer would say that the ball is always located somewhere between 0° and 360°. Thus it is common practice to subtract an integer number of 2π rad, representing the completed revolutions. Because $20.94/2\pi = 3.333$, we can write

$$\begin{aligned}\theta_f &= 20.94\text{ rad} = 3.333 \times 2\pi\text{ rad}\\ &= 3 \times 2\pi\text{ rad} + 0.333 \times 2\pi\text{ rad}\\ &= 3 \times 2\pi\text{ rad} + 2.09\text{ rad}\end{aligned}$$

In other words, at $t = 2.0$ s the ball has completed 3 rev and is 2.09 rad = 120° into its fourth revolution. An observer would say that the ball's position is $\theta_f = 120°$.

As Figure 4.21 showed, the velocity vector $\vec{v}$ is always tangent to the circle. In other words, the velocity vector has only a *tangential component,* which we will designate v_t. The tangential velocity is positive for ccw motion, negative for cw motion.

Combining $v = 2\pi r/T$ for the speed with $\omega = 2\pi/T$ for the angular velocity—but keeping the sign of ω to indicate the direction of motion—we see that the tangential velocity and the angular velocity are related by

$$v_t = \omega r \qquad \text{(with } \omega \text{ in rad/s)} \tag{4.28}$$

Because v_t is the only nonzero component of $\vec{v}$, the particle's speed is $v = |v_t| = |\omega|r$. We'll sometimes write this as $v = \omega r$ if there's no ambiguity about the sign of ω.

NOTE While it may be convenient in some problems to measure ω in rev/s or rpm, you must convert to SI units of rad/s before using Equation 4.28.

As a simple example, a particle moving cw at 2.0 m/s in a circle of radius 40 cm has angular velocity

$$\omega = \frac{v_t}{r} = \frac{-2.0 \text{ m/s}}{0.40 \text{ m}} = -5.0 \text{ rad/s}$$

where v_t and ω are negative because the motion is clockwise. Notice the units. Velocity divided by distance has units of s^{-1}. But because the division, in this case, gives us an angular quantity, we've inserted the *dimensionless* unit rad to give ω the appropriate units of rad/s.

STOP TO THINK 4.5 A particle moves cw around a circle at constant speed for 2.0 s. It then reverses direction and moves ccw at half the original speed until it has traveled through the same angle. Which is the particle's angle-versus-time graph?

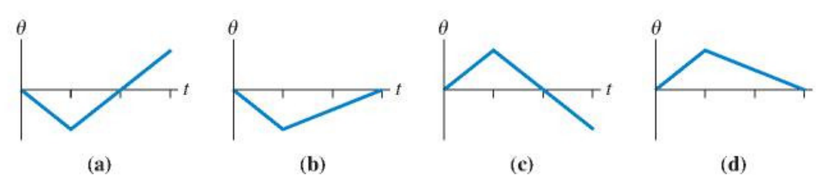
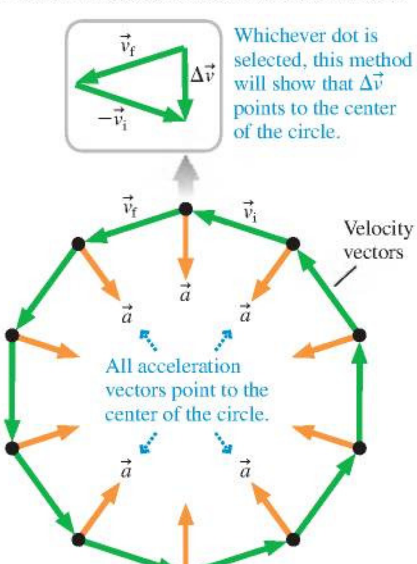


FIGURE 4.27 Using Tactics Box 4.1 to find Maria's acceleration on the Ferris wheel.



Maria's acceleration is an acceleration of changing direction, not of changing speed.

4.5 Centripetal Acceleration

FIGURE 4.27 shows a motion diagram of Maria riding a Ferris wheel at the amusement park. Maria has constant speed but *not* constant velocity because her velocity vector is changing direction. She may not be speeding up, but Maria *is* accelerating because her velocity is changing. The inset to Figure 4.27 applies the rules of Tactics Box 4.1 to find that—at every point—**Maria's acceleration vector points toward the center of the circle.** This is an acceleration due to changing direction rather than changing speed. Because the instantaneous velocity is tangent to the circle, $\vec{v}$ and $\vec{a}$ are perpendicular to each other at all points on the circle.

The acceleration of uniform circular motion is called **centripetal acceleration,** a term from a Greek root meaning "center seeking." Centripetal acceleration is not a new type of acceleration; all we are doing is *naming* an acceleration that corresponds to a particular type of motion. The magnitude of the centripetal acceleration is constant because each successive $\Delta\vec{v}$ in the motion diagram has the same length.

The motion diagram tells us the direction of $\vec{a}$, but it doesn't give us a value for a. To complete our description of uniform circular motion, we need to find a quantitative relationship between a and the particle's speed v. FIGURE 4.28 shows the velocity $\vec{v}_i$

at one instant of motion and the velocity $\vec{v}_f$ an infinitesimal amount of time dt later. During this small interval of time, the particle has moved through the infinitesimal angle $d\theta$ and traveled distance $ds = r\,d\theta$.

By definition, the acceleration is $\vec{a} = d\vec{v}/dt$. We can see from the inset to Figure 4.28 that $d\vec{v}$ points toward the center of the circle—that is, $\vec{a}$ is a centripetal acceleration. To find the magnitude of $\vec{a}$, we can see from the isosceles triangle of velocity vectors that, if $d\theta$ is in radians,

$$dv = |d\vec{v}| = v\,d\theta \tag{4.29}$$

For uniform circular motion at constant speed, $v = ds/dt = r\,d\theta/dt$ and thus the time to rotate through angle $d\theta$ is

$$dt = \frac{r\,d\theta}{v} \tag{4.30}$$

Combining Equations 4.29 and 4.30, we see that the acceleration has magnitude

$$a = |\vec{a}| = \frac{|d\vec{v}|}{dt} = \frac{v\,d\theta}{r\,d\theta/v} = \frac{v^2}{r}$$

In vector notation, we can write

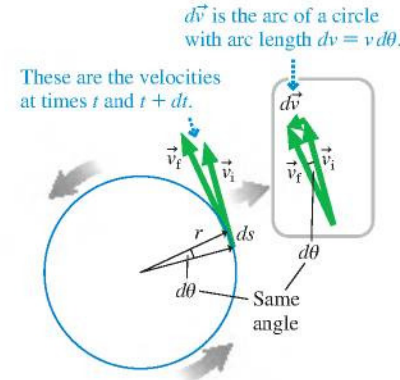
$$\vec{a} = \left(\frac{v^2}{r}, \text{ toward center of circle}\right) \qquad \text{(centripetal acceleration)} \tag{4.31}$$

Using Equation 4.28, $v = \omega r$, we can also express the magnitude of the centripetal acceleration in terms of the angular velocity ω as

$$a = \omega^2 r \tag{4.32}$$

NOTE Centripetal acceleration is not a constant acceleration. The magnitude of the centripetal acceleration is constant during uniform circular motion, but the direction of $\vec{a}$ is constantly changing. **Thus the constant-acceleration kinematics equations of Chapter 2 do *not* apply to circular motion.**

FIGURE 4.28 Finding the acceleration of circular motion.



The Uniform Circular Motion Model

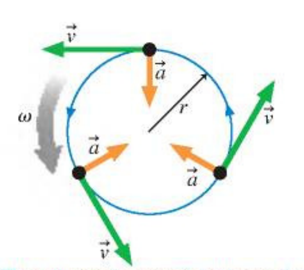
The **uniform circular motion model** is especially important because it applies not only to particles moving in circles but also to the uniform rotation of solid objects.

MODEL 4.2

Uniform circular motion

For motion with constant angular velocity ω.

- Applies to a particle moving along a circular trajectory at constant speed or to points on a solid object rotating at a steady rate.
- Mathematically:
  - The tangential velocity is $v_t = \omega r$.
  - The centripetal acceleration is v^2/r or $\omega^2 r$.
  - ω and v_t are positive for ccw rotation, negative for cw rotation.
- Limitations: Model fails if rotation isn't steady.



Exercise 20

EXAMPLE 4.12 The acceleration of a Ferris wheel

A typical carnival Ferris wheel has a radius of 9.0 m and rotates 4.0 times per minute. What speed and acceleration do the riders experience?

MODEL Model the rider as a particle in uniform circular motion.

SOLVE The period is $T = \frac{1}{4}$ min = 15 s. From Equation 4.21, a rider's speed is

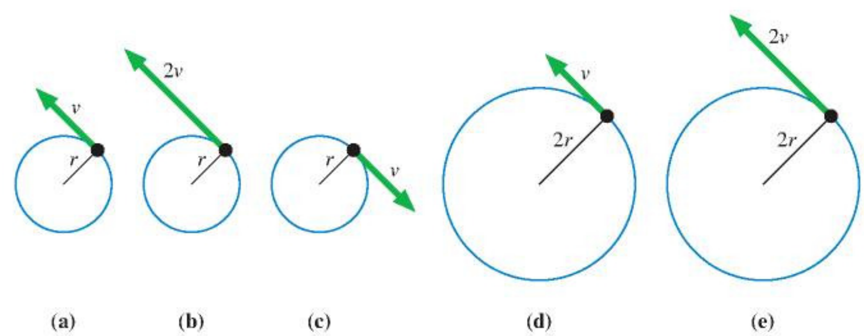
$$v = \frac{2\pi r}{T} = \frac{2\pi(9.0\text{ m})}{15\text{ s}} = 3.77\text{ m/s}$$

Consequently, the centripetal acceleration has magnitude

$$a = \frac{v^2}{r} = \frac{(3.77\text{ m/s})^2}{9.0\text{ m}} = 1.6\text{ m/s}^2$$

ASSESS This was not intended to be a profound problem, merely to illustrate how centripetal acceleration is computed. The acceleration is enough to be noticed and make the ride interesting, but not enough to be scary.

STOP TO THINK 4.6 Rank in order, from largest to smallest, the centripetal accelerations a_a to a_e of particles a to e.

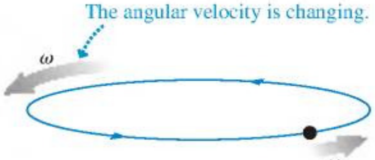


4.6 Nonuniform Circular Motion

A roller coaster car doing a loop-the-loop slows down as it goes up one side, speeds up as it comes back down the other. The ball in a roulette wheel gradually slows until it stops. Circular motion with a changing speed is called **nonuniform circular motion.** As you'll see, nonuniform circular motion is analogous to accelerated linear motion.

FIGURE 4.29 shows a point speeding up as it moves around a circle. This might be a car speeding up around a curve or simply a point on a solid object that is rotating faster and faster. The key feature of the motion is a *changing angular velocity*. For linear motion, we defined acceleration as $a_x = dv_x/dt$. By analogy, let's define the **angular acceleration** α (Greek alpha) of a rotating object, or a point on the object, to be

FIGURE 4.29 Circular motion with a changing angular velocity.



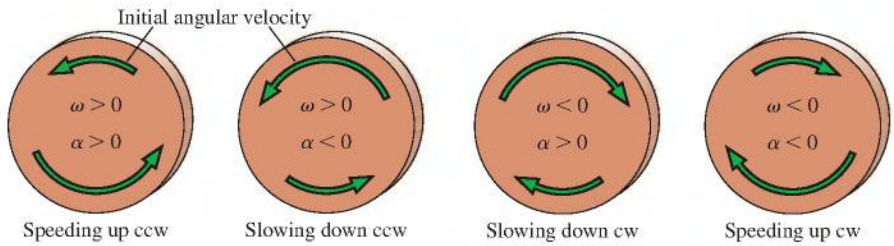
$$\alpha \equiv \frac{d\omega}{dt} \quad \text{(angular acceleration)} \tag{4.33}$$

Angular acceleration is the *rate* at which the angular velocity ω changes, just as linear acceleration is the rate at which the linear velocity v_x changes. The units of angular acceleration are rad/s^2.

For linear acceleration, you learned that a_x and v_x have the same sign when an object is speeding up, opposite signs when it is slowing down. The same rule applies to circular and rotational motion: ω and α have the same sign when the rotation is speeding up, opposite signs if it is slowing down. These ideas are illustrated in FIGURE 4.30.

NOTE Be careful with the sign of α. You learned in Chapter 2 that positive and negative values of the acceleration can't be interpreted as simply "speeding up" and "slowing down." Similarly, positive and negative values of angular acceleration can't be interpreted as a rotation that is speeding up or slowing down.

FIGURE 4.30 The signs of angular velocity and acceleration. The rotation is speeding up if ω and α have the same sign, slowing down if they have opposite signs.



Angular position, angular velocity, and angular acceleration are defined exactly the same as linear position, velocity, and acceleration—simply starting with an angular rather than a linear measurement of position. Consequently, **the graphical interpretation and the kinematic equations of circular/rotational motion with constant angular acceleration are exactly the same as for linear motion with constant acceleration.** This is shown in the **constant angular acceleration model** below. All the problem-solving techniques you learned in Chapter 2 for linear motion carry over to circular and rotational motion.

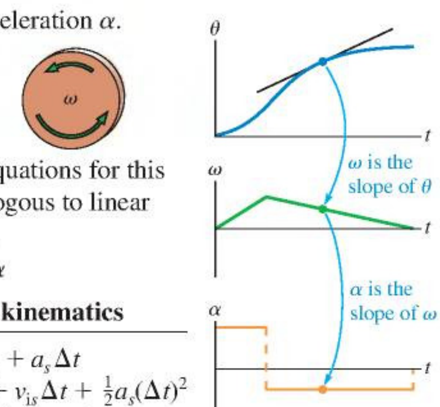
MODEL 4.3

Constant angular acceleration

For motion with constant angular acceleration α.

- Applies to particles with circular trajectories and to rotating solid objects.
- Mathematically: The graphs and equations for this circular/rotational motion are analogous to linear motion with constant acceleration.
  - Analogs: $s \rightarrow \theta \quad v_s \rightarrow \omega \quad a_s \rightarrow \alpha$

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha\Delta t$	$v_{fs} = v_{is} + a_s\Delta t$
$\theta_f = \theta_i + \omega_i\Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is}\Delta t + \frac{1}{2}a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$	$v_{fs}^2 = v_{is}^2 + 2a_s\Delta s$

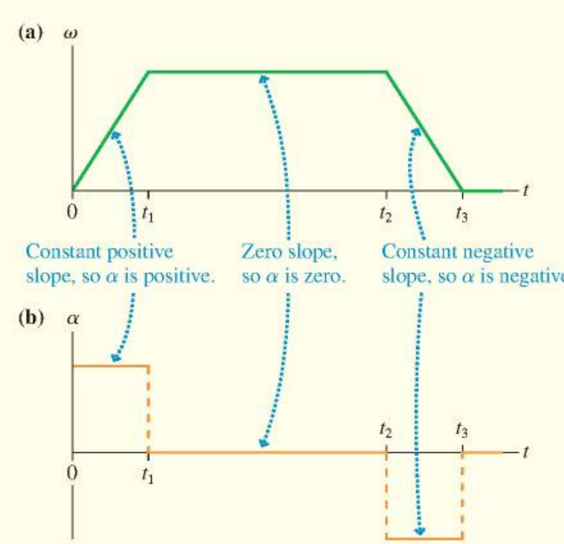


EXAMPLE 4.13 A rotating wheel

FIGURE 4.31a is a graph of angular velocity versus time for a rotating wheel. Describe the motion and draw a graph of angular acceleration versus time.

SOLVE This is a wheel that starts from rest, gradually speeds up *counterclockwise* until reaching top speed at t_1, maintains a constant angular velocity until t_2, then gradually slows down until stopping at t_3. The motion is always ccw because ω is always positive. The angular acceleration graph of FIGURE 4.32b is based on the fact that α is the slope of the ω-versus-t graph.

Conversely, the initial linear increase of ω can be seen as the increasing area under the α-versus-t graph as t increases from 0 to t_1. The angular velocity doesn't change from t_1 to t_2 when the area under the α-versus-t is zero.



► FIGURE 4.31 ω-versus-t graph and the corresponding α-versus-t graph for a rotating wheel.

EXAMPLE 4.14 | A slowing fan

A ceiling fan spinning at 60 rpm coasts to a stop 25 s after being turned off. How many revolutions does it make while stopping?

MODEL Model the fan as a rotating object with constant angular acceleration.

SOLVE We don't know which direction the fan is rotating, but the fact that the rotation is slowing tells us that ω and α have opposite signs. We'll assume that ω is positive. We need to convert the initial angular velocity to SI units:

$$\omega_i = 60\,\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ s}} \times \frac{2\pi\text{ rad}}{1\text{ rev}} = 6.28\text{ rad/s}$$

We can use the first rotational kinematics equation in Model 4.3 to find the angular acceleration:

$$\alpha = \frac{\omega_f - \omega_i}{\Delta t} = \frac{0\text{ rad/s} - 6.28\text{ rad/s}}{25\text{ s}} = -0.25\text{ rad/s}^2$$

Then, from the second rotational kinematic equation, the angular displacement during these 25 s is

$$\begin{aligned}\Delta\theta &= \omega_i\,\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2\\ &= (6.28\text{ rad/s})(25\text{ s}) + \tfrac{1}{2}(-0.25\text{ rad/s}^2)(25\text{ s})^2\\ &= 78.9\text{ rad} \times \frac{1\text{ rev}}{2\pi\text{ rad}} = 13\text{ rev}\end{aligned}$$

The kinematic equation returns an angle in rad, but the question asks for revolutions, so the last step was a unit conversion.

ASSESS Turning through 13 rev in 25 s while stopping seems reasonable. Notice that the problem is solved just like the linear kinematics problems you learned to solve in Chapter 2.

Tangential Acceleration

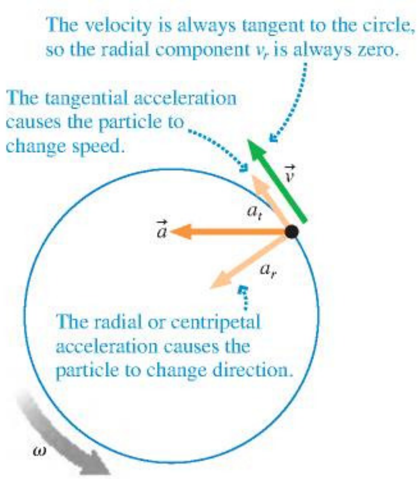


FIGURE 4.32 Acceleration in nonuniform circular motion.

FIGURE 4.32 shows a particle in nonuniform circular motion. *Any* circular motion, whether uniform or nonuniform, has a centripetal acceleration because the particle is changing direction; this was the acceleration component $\vec{a}_\perp$ of Figure 4.6. As a vector component, the centripetal acceleration, which points radially toward the center of the circle, is the **radial acceleration** a_r. The expression $a_r = v_t^2/r = \omega^2 r$ is still valid in nonuniform circular motion.

For a particle to speed up or slow down as it moves around a circle, it needs—in addition to the centripetal acceleration—an acceleration parallel to the trajectory or, equivalently, parallel to $\vec{v}$. This is the acceleration component $\vec{a}_\parallel$ associated with changing speed. We'll call this the **tangential acceleration** a_t because, like the velocity v_t, it is always tangent to the circle. Because of the tangential acceleration, **the acceleration vector $\vec{a}$ of a particle in nonuniform circular motion does not point toward the center of the circle**. It points "ahead" of center for a particle that is speeding up, as in Figure 4.32, but it would point "behind" center for a particle slowing down. You can see from Figure 4.32 that the magnitude of the acceleration is

$$a = \sqrt{a_r^2 + a_t^2} \tag{4.34}$$

If a_t is constant, then the arc length s traveled by the particle around the circle and the tangential velocity v_t are found from constant-acceleration kinematics:

$$\begin{aligned}s_f &= s_i + v_{it}\,\Delta t + \tfrac{1}{2}a_t(\Delta t)^2\\ v_{ft} &= v_{it} + a_t\,\Delta t\end{aligned} \tag{4.35}$$

Because tangential acceleration is the rate at which the tangential velocity changes, $a_t = dv_t/dt$, and we already know that the tangential velocity is related to the angular velocity by $v_t = \omega r$, it follows that

$$a_t = \frac{dv_t}{dt} = \frac{d(\omega r)}{dt} = \frac{d\omega}{dt}r = \alpha r \tag{4.36}$$

Thus $v_t = \omega r$ and $a_t = \alpha r$ are analogous equations for the tangential velocity and acceleration. In Example 4.14, where we found the fan to have angular acceleration $\alpha = -0.25\text{ rad/s}^2$, a blade tip 65 cm from the center would have tangential acceleration

$$a_t = \alpha r = (-0.25\text{ rad/s}^2)(0.65\text{ m}) = -0.16\text{ m/s}^2$$

EXAMPLE 4.15 Analyzing rotational data

You've been assigned the task of measuring the start-up characteristics of a large industrial motor. After several seconds, when the motor has reached full speed, you know that the angular acceleration will be zero, but you hypothesize that the angular acceleration may be constant during the first couple of seconds as the motor speed increases. To find out, you attach a shaft encoder to the 3.0-cm-diameter axle. A shaft encoder is a device that converts the angular position of a shaft or axle to a signal that can be read by a computer. After setting the computer program to read four values a second, you start the motor and acquire the following data:

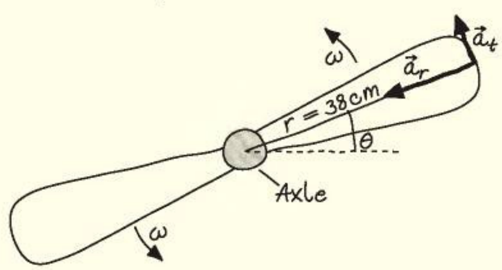
Time (s)	Angle (°)	Time (s)	Angle (°)
0.00	0	1.00	267
0.25	16	1.25	428
0.50	69	1.50	620
0.75	161		

a. Do the data support your hypothesis of a constant angular acceleration? If so, what is the angular acceleration? If not, is the angular acceleration increasing or decreasing with time?
b. A 76-cm-diameter blade is attached to the motor shaft. At what time does the acceleration of the tip of the blade reach 10 m/s^2?

MODEL The axle is rotating with nonuniform circular motion. Model the tip of the blade as a particle.

VISUALIZE FIGURE 4.33 shows that the blade tip has both a tangential and a radial acceleration.

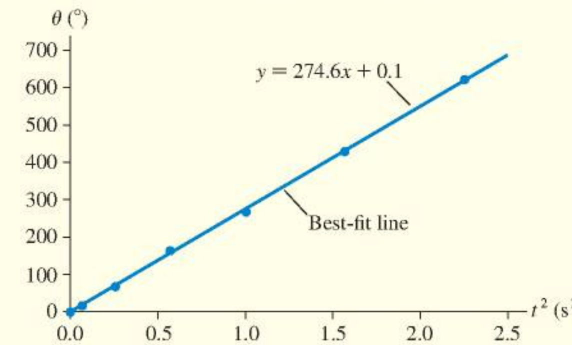
FIGURE 4.33 Pictorial representation of the axle and blade.



SOLVE a. *If* the motor starts up with constant angular acceleration, with $\theta_i = 0$ and $\omega_i = 0$ rad/s, the angle-time equation of rotational kinematics is $\theta = \frac{1}{2}\alpha t^2$. This can be written as a linear equation $y = mx + b$ if we let $\theta = y$ and $t^2 = x$. That is, constant angular acceleration predicts that a graph of θ versus t^2 should be a straight line with slope $m = \frac{1}{2}\alpha$ and y-intercept $b = 0$. We can test this.

FIGURE 4.34 is the graph of θ versus t^2, and it confirms our hypothesis that the motor starts up with constant angular acceleration. The best-fit line, found using a spreadsheet, gives a slope of 274.6°/s^2. The units come not from the spreadsheet but by looking at the units of rise (°) over run (s^2 because we're graphing t^2 on the x-axis). Thus the angular acceleration is

FIGURE 4.34 Graph of θ versus t^2 for the motor shaft.



$$\alpha = 2m = 549.2°/\text{s}^2 \times \frac{\pi \text{ rad}}{180°} = 9.6 \text{ rad/s}^2$$

where we used 180° = π rad to convert to SI units of rad/s^2.

c. The magnitude of the linear acceleration is

$$a = \sqrt{a_r^2 + a_t^2}$$

The tangential acceleration of the blade tip is

$$a_t = \alpha r = (9.6 \text{ rad/s}^2)(0.38 \text{ m}) = 3.65 \text{ m/s}^2$$

We were careful to use the blade's radius, not its diameter, and we kept an extra significant figure to avoid round-off error. The radial (centripetal) acceleration increases as the rotation speed increases, and the total acceleration reaches 10 m/s^2 when

$$a_r = \sqrt{a^2 - a_t^2} = \sqrt{(10 \text{ m/s}^2)^2 - (3.65 \text{ m/s}^2)^2} = 9.31 \text{ m/s}^2$$

Radial acceleration is $a_r = \omega^2 r$, so the corresponding angular velocity is

$$\omega = \sqrt{\frac{a_r}{r}} = \sqrt{\frac{9.31 \text{ m/s}^2}{0.38 \text{ m}}} = 4.95 \text{ rad/s}$$

For constant angular acceleration, $\omega = \alpha t$, so this angular velocity is achieved at

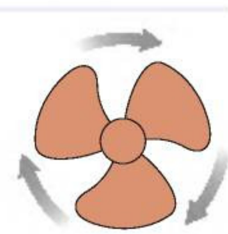
$$t = \frac{\omega}{\alpha} = \frac{4.95 \text{ rad/s}}{9.6 \text{ rad/s}^2} = 0.52 \text{ s}$$

Thus it takes 0.52 s for the acceleration of the blade tip to reach 10 m/s^2.

ASSESS The acceleration at the tip of a long blade is likely to be large. It seems plausible that the acceleration would reach 10 m/s^2 in $\approx$ 0.5 s.

STOP TO THINK 4.7 The fan blade is slowing down. What are the signs of ω and α?

a. ω is positive and α is positive.
b. ω is positive and α is negative.
c. ω is negative and α is positive.
d. ω is negative and α is negative.



CHALLENGE EXAMPLE 4.16 Hit the target!

One day when you come into lab, you see a spring-loaded wheel that can launch a ball straight up. To do so, you place the ball in a cup on the rim of the wheel, turn the wheel to stretch the spring, then release. The wheel rotates through an angle $\Delta\theta$, then hits a stop when the cup is level with the axle and pointing straight up. The cup stops, but the ball flies out and keeps going. You're told that the wheel has been designed to have constant angular acceleration as it rotates through $\Delta\theta$. The lab assignment is first to measure the wheel's angular acceleration. Then the lab instructor is going to place a target at height h above the point where the ball is launched. Your task will be to launch the ball so that it just barely hits the target.

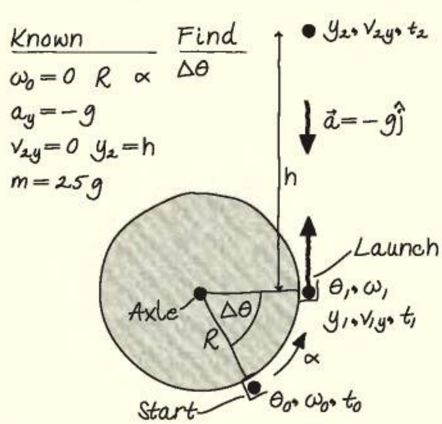
a. Find an expression in terms of quantities that you can measure for the angle $\Delta\theta$ that launches the ball at the correct speed.

b. Evaluate $\Delta\theta$ if the wheel's diameter is 62 cm, you've determined that its angular acceleration is 200 rad/s^2, the mass of the ball is 25 g, and the instructor places the target 190 cm above the launch point.

MODEL Model the ball as a particle. It first undergoes circular motion that we'll model as having constant angular acceleration. We'll then ignore air resistance and model the vertical motion as free fall.

VISUALIZE FIGURE 4.35 is a pictorial representation. This is a two-part problem, with the speed at the end of the angular acceleration being the launch speed for the vertical motion. We've chosen to call the wheel radius R and the target height h. These and the angular acceleration α are considered "known" because we will measure them, but we don't have numerical values at this time.

FIGURE 4.35 Pictorial representation of the ball launcher.



SOLVE

a. The circular motion problem and the vertical motion problem are connected through the ball's speed: The final speed of the angular acceleration is the launch speed of the vertical motion. We don't know anything about time intervals, which suggests using the kinematic equations that relate distance and acceleration (for the vertical motion) and angle and angular acceleration (for the circular motion). For the angular acceleration, with $\omega_0 = 0$ rad/s,

$$\omega_1^2 = \omega_0^2 + 2\alpha\,\Delta\theta = 2\alpha\,\Delta\theta$$

The final speed of the ball and cup, when the wheel hits the stop, is

$$v_1 = \omega_1 R = R\sqrt{2\alpha\,\Delta\theta}$$

Thus the vertical-motion problem begins with the ball being shot upward with velocity $v_{1y} = R\sqrt{2\alpha\,\Delta\theta}$. How high does it go? The highest point is the point where $v_{2y} = 0$, so the free-fall equation is

$$v_{2y}^2 = 0 = v_{1y}^2 - 2g\,\Delta y = R^2 \cdot 2\alpha\,\Delta\theta - 2gh$$

Rather than solve for height h, we need to solve for the angle that produces a given height. This is

$$\Delta\theta = \frac{gh}{\alpha R^2}$$

Once we've determined the properties of the wheel and then measured the height at which our instructor places the target, we'll quickly be able to calculate the angle through which we should pull back the wheel to launch the ball.

b. For the values given in the problem statement, $\Delta\theta = 0.969$ rad $= 56°$. Don't forget that equations involving angles need values in radians and return values in radians.

ASSESS The angle needed to be less than 90° or else the ball would fall out of the cup before launch. And an angle of only a few degrees would seem suspiciously small. Thus 56° seems to be reasonable. Notice that the mass was not needed in this problem. Part of becoming a better problem solver is evaluating the information you have to see what is relevant. Some homework problems will help you develop this skill by providing information that isn't necessary.

SUMMARY

The goal of Chapter 4 has been to learn how to solve problems about motion in a plane.

GENERAL PRINCIPLES

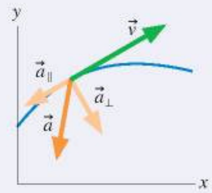
The **instantaneous velocity**

$$\vec{v} = d\vec{r}/dt$$

is a vector tangent to the trajectory.

The **instantaneous acceleration** is

$$\vec{a} = d\vec{v}/dt$$



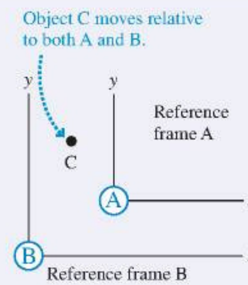
$\vec{a}_\parallel$, the component of $\vec{a}$ parallel to $\vec{v}$, is responsible for change of *speed*. $\vec{a}_\perp$, the component of $\vec{a}$ perpendicular to $\vec{v}$, is responsible for change of *direction*.

Relative Motion

If object C moves relative to reference frame A with velocity $\vec{v}_{CA}$, then it moves relative to a different reference frame B with velocity

$$\vec{v}_{CB} = \vec{v}_{CA} + \vec{v}_{AB}$$

where $\vec{v}_{AB}$ is the velocity of A relative to B. This is the Galilean transformation of velocity.



IMPORTANT CONCEPTS

Uniform Circular Motion

Angular velocity $\omega = d\theta/dt$.

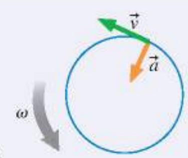
v_t and ω are constant:

$$v_t = \omega r$$

The centripetal acceleration points toward the center of the circle:

$$a = \frac{v^2}{r} = \omega^2 r$$

It changes the particle's direction but not its speed.



Nonuniform Circular Motion

Angular acceleration $\alpha = d\omega/dt$.

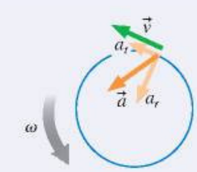
The radial acceleration

$$a_r = \frac{v^2}{r} = \omega^2 r$$

changes the particle's direction. The tangential component

$$a_t = \alpha r$$

changes the particle's speed.



APPLICATIONS

Kinematics in two dimensions

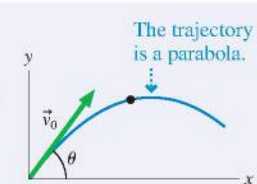
If $\vec{a}$ is constant, then the x- and y-components of motion are independent of each other.

$$x_f = x_i + v_{ix}\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$
$$y_f = y_i + v_{iy}\,\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$
$$v_{fx} = v_{ix} + a_x\,\Delta t$$
$$v_{fy} = v_{iy} + a_y\,\Delta t$$

Projectile motion is motion under the influence of only gravity.

MODEL Model as a particle launched with speed v_0 at angle θ.

VISUALIZE Use coordinates with the x-axis horizontal and the y-axis vertical.

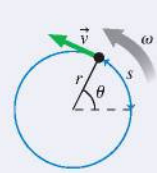


SOLVE The horizontal motion is uniform with $v_x = v_0\cos\theta$. The vertical motion is free fall with $a_y = -g$. The x and y kinematic equations have the *same* value for Δt.

Circular motion kinematics

Period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\omega}$

Angular position $\theta = \dfrac{s}{r}$



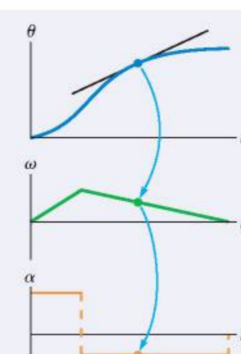
Constant angular acceleration

$$\omega_f = \omega_i + \alpha\,\Delta t$$
$$\theta_f = \theta_i + \omega_i\,\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$
$$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\theta$$

Circular motion graphs and kinematics are analogous to linear motion with constant acceleration.

Angle, angular velocity, and angular acceleration are related graphically.

- The angular velocity is the slope of the angular position graph.
- The angular acceleration is the slope of the angular velocity graph.



TERMS AND NOTATION

projectile	uniform circular motion	angular velocity, ω	angular acceleration, α
launch angle, θ	period, T	centripetal acceleration	constant angular acceleration model
projectile motion model	angular position, θ	uniform circular motion model	radial acceleration, a_r
reference frame	arc length, s	nonuniform circular motion	tangential acceleration, a_t
Galilean transformation of velocity	radians		
	angular displacement, $\Delta\theta$		

CONCEPTUAL QUESTIONS

1. a. At this instant, is the particle in **FIGURE Q4.1** speeding up, slowing down, or traveling at constant speed?
   b. Is this particle curving to the right, curving to the left, or traveling straight?

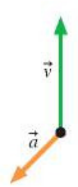


FIGURE Q4.1

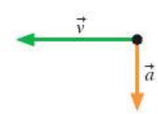


FIGURE Q4.2

2. a. At this instant, is the particle in **FIGURE Q4.2** speeding up, slowing down, or traveling at constant speed?
   b. Is this particle curving upward, curving downward, or traveling straight?
3. Tarzan swings through the jungle by hanging from a vine.
   a. Immediately after stepping off a branch to swing over to another tree, is Tarzan's acceleration $\vec{a}$ zero or not zero? If not zero, which way does it point? Explain.
   b. Answer the same question at the lowest point in Tarzan's swing.
4. A projectile is launched at an angle of 30°.
   a. Is there any point on the trajectory where $\vec{v}$ and $\vec{a}$ are parallel to each other? If so, where?
   b. Is there any point where $\vec{v}$ and $\vec{a}$ are perpendicular to each other? If so, where?
5. For a projectile, which of the following quantities are constant during the flight: x, y, r, v_x, v_y, v, a_x, a_y? Which of these quantities are zero throughout the flight?
6. A cart that is rolling at constant velocity on a level table fires a ball straight up.
   a. When the ball comes back down, will it land in front of the launching tube, behind the launching tube, or directly in the tube? Explain.
   b. Will your answer change if the cart is accelerating in the forward direction? If so, how?
7. A rock is thrown from a bridge at an angle 30° below horizontal. Immediately after the rock is released, is the magnitude of its acceleration greater than, less than, or equal to g? Explain.
8. Anita is running to the right at 5 m/s in **FIGURE Q4.8**. Balls 1 and 2 are thrown toward her by friends standing on the ground. According to Anita, both balls are approaching her at 10 m/s. Which ball was thrown at a faster speed? Or were they thrown with the same speed? Explain.

FIGURE Q4.8

9. An electromagnet on the ceiling of an airplane holds a steel ball. When a button is pushed, the magnet releases the ball. The experiment is first done while the plane is parked on the ground, and the point where the ball hits the floor is marked with an X. Then the experiment is repeated while the plane is flying level at a steady 500 mph. Does the ball land slightly in front of the X (toward the nose of the plane), on the X, or slightly behind the X (toward the tail of the plane)? Explain.
10. Zack is driving past his house in **FIGURE Q4.10**. He wants to toss his physics book out the window and have it land in his driveway. If he lets go of the book exactly as he passes the end of the driveway, should he direct his throw outward and toward the front of the car (throw 1), straight outward (throw 2), or outward and toward the back of the car (throw 3)? Explain.

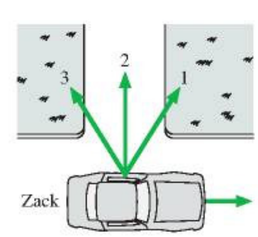


FIGURE Q4.10

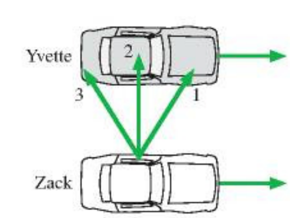


FIGURE Q4.11

11. In **FIGURE Q4.11**, Yvette and Zack are driving down the freeway side by side with their windows down. Zack wants to toss his physics book out the window and have it land in Yvette's front seat. Ignoring air resistance, should he direct his throw outward and toward the front of the car (throw 1), straight outward (throw 2), or outward and toward the back of the car (throw 3)? Explain.

12. In uniform circular motion, which of the following quantities are constant: speed, instantaneous velocity, the tangential component of velocity, the radial component of acceleration, the tangential component of acceleration? Which of these quantities are zero throughout the motion?

13. FIGURE Q4.13 shows three points on a steadily rotating wheel.
   a. Rank in order, from largest to smallest, the angular velocities ω_1, ω_2, and ω_3 of these points. Explain.
   b. Rank in order, from largest to smallest, the speeds v_1, v_2, and v_3 of these points. Explain.

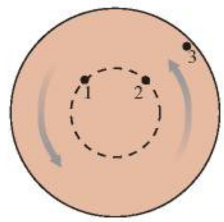


FIGURE Q4.13

14. FIGURE Q4.14 shows four rotating wheels. For each, determine the signs (+ or −) of ω and α.

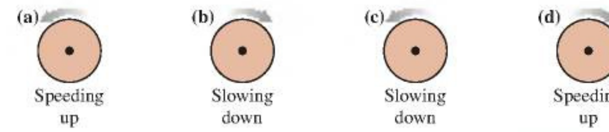


FIGURE Q4.14

15. FIGURE Q4.15 shows a pendulum at one end point of its arc.
   a. At this point, is ω positive, negative, or zero? Explain.
   b. At this point, is α positive, negative, or zero? Explain.


FIGURE Q4.15

EXERCISES AND PROBLEMS

Exercises

Section 4.1 Motion in Two Dimensions

Problems 1 and 2 show a partial motion diagram. For each:
   a. Complete the motion diagram by adding acceleration vectors.
   b. Write a physics *problem* for which this is the correct motion diagram. Be imaginative! Don't forget to include enough information to make the problem complete and to state clearly what is to be found.

1. |

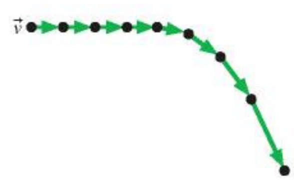


FIGURE EX4.1

2. |

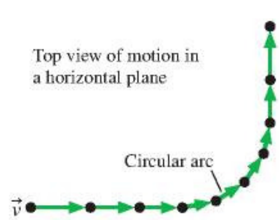
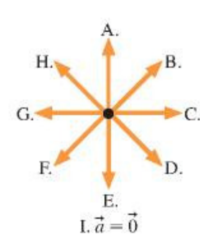


FIGURE EX4.2

Answer Problems 3 through 5 by choosing one of the eight labeled acceleration vectors or selecting option I: $\vec{a} = \vec{0}$.



3. || At this instant, the particle has steady speed and is curving to the right. What is the direction of its acceleration?

FIGURE EX4.3

4. || At this instant, the particle is speeding up and curving upward. What is the direction of its acceleration?

FIGURE EX4.4

5. || At this instant, the particle is speeding up and curving downward. What is the direction of its acceleration?

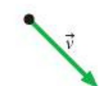


FIGURE EX4.5

6. || A rocket-powered hockey puck moves on a horizontal frictionless table. FIGURE EX4.6 shows graphs of v_x and v_y, the x- and y-components of the puck's velocity. The puck starts at the origin.
   a. In which direction is the puck moving at $t = 2$ s? Give your answer as an angle from the x-axis.
   b. How far from the origin is the puck at $t = 5$ s?

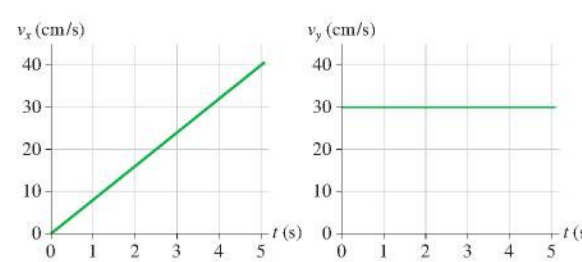


FIGURE EX4.6

7. || A rocket-powered hockey puck moves on a horizontal frictionless table. FIGURE EX4.7 shows graphs of v_x and v_y, the x- and y-components of the puck's velocity. The puck starts at the origin. What is the magnitude of the puck's acceleration at $t = 5$ s?

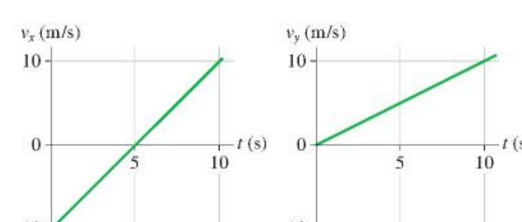


FIGURE EX4.7

8. ‖ A particle's trajectory is described by $x = \left(\frac{1}{2}t^3 - 2t^2\right)$ m and $y = \left(\frac{1}{2}t^2 - 2t\right)$ m, where t is in s.
CALC
   a. What are the particle's position and speed at $t = 0$ s and $t = 4$ s?
   b. What is the particle's direction of motion, measured as an angle from the x-axis, at $t = 0$ s and $t = 4$ s?

9. | A particle moving in the xy-plane has velocity $\vec{v} = (2t\hat{\imath} + (3 - t^2)\hat{\jmath})$ m/s, where t is in s. What is the particle's acceleration vector at $t = 4$ s?
CALC

10. ‖ You have a remote-controlled car that has been programmed to have velocity $\vec{v} = (-3t\hat{\imath} + 2t^2\hat{\jmath})$ m/s, where t is in s. At $t = 0$ s, the car is at $\vec{r}_0 = (3.0\hat{\imath} + 2.0\hat{\jmath})$ m. What are the car's (a) position vector and (b) acceleration vector at $t = 2.0$ s?
CALC

Section 4.2 Projectile Motion

11. ‖ A ball thrown horizontally at 25 m/s travels a horizontal distance of 50 m before hitting the ground. From what height was the ball thrown?

12. | A physics student on Planet Exidor throws a ball, and it follows the parabolic trajectory shown in FIGURE EX4.12. The ball's position is shown at 1 s intervals until $t = 3$ s. At $t = 1$ s, the ball's velocity is $\vec{v} = (2.0\,\hat{\imath} + 2.0\,\hat{\jmath})$ m/s.
   a. Determine the ball's velocity at $t = 0$ s, 2 s, and 3 s.
   b. What is the value of g on Planet Exidor?
   c. What was the ball's launch angle?

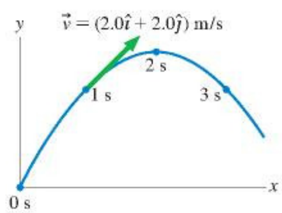


FIGURE EX4.12

13. ‖ A supply plane needs to drop a package of food to scientists working on a glacier in Greenland. The plane flies 100 m above the glacier at a speed of 150 m/s. How far short of the target should it drop the package?

14. ‖ A rifle is aimed horizontally at a target 50 m away. The bullet hits the target 2.0 cm below the aim point.
   a. What was the bullet's flight time?
   b. What was the bullet's speed as it left the barrel?

15. ‖ In the Olympic shotput event, an athlete throws the shot with an initial speed of 12.0 m/s at a 40.0° angle from the horizontal. The shot leaves her hand at a height of 1.80 m above the ground. How far does the shot travel?

16. ‖ On the Apollo 14 mission to the moon, astronaut Alan Shepard hit a golf ball with a 6 iron. The free-fall acceleration on the moon is 1/6 of its value on earth. Suppose he hit the ball with a speed of 25 m/s at an angle 30° above the horizontal.
   a. How much farther did the ball travel on the moon than it would have on earth?
   b. For how much more time was the ball in flight?

17. ||| A baseball player friend of yours wants to determine his pitching speed. You have him stand on a ledge and throw the ball horizontally from an elevation 4.0 m above the ground. The ball lands 25 m away. What is his pitching speed?

Section 4.3 Relative Motion

18. ‖ A boat takes 3.0 hours to travel 30 km down a river, then 5.0 hours to return. How fast is the river flowing?

19. ‖ When the moving sidewalk at the airport is broken, as it often seems to be, it takes you 50 s to walk from your gate to baggage claim. When it is working and you stand on the moving sidewalk the entire way, without walking, it takes 75 s to travel the same distance. How long will it take you to travel from the gate to baggage claim if you walk while riding on the moving sidewalk?

20. | Mary needs to row her boat across a 100-m-wide river that is flowing to the east at a speed of 1.0 m/s. Mary can row with a speed of 2.0 m/s.
   a. If Mary points her boat due north, how far from her intended landing spot will she be when she reaches the opposite shore?
   b. What is her speed with respect to the shore?

21. | A kayaker needs to paddle north across a 100-m-wide harbor. The tide is going out, creating a tidal current that flows to the east at 2.0 m/s. The kayaker can paddle with a speed of 3.0 m/s.
   a. In which direction should he paddle in order to travel straight across the harbor?
   b. How long will it take him to cross?

22. ‖ Susan, driving north at 60 mph, and Trent, driving east at 45 mph, are approaching an intersection. What is Trent's speed relative to Susan's reference frame?

Section 4.4 Uniform Circular Motion

23. ‖ FIGURE EX4.23 shows the angular-velocity-versus-time graph for a particle moving in a circle. How many revolutions does the object make during the first 4 s?

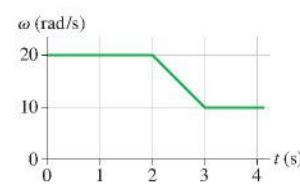


FIGURE EX4.23

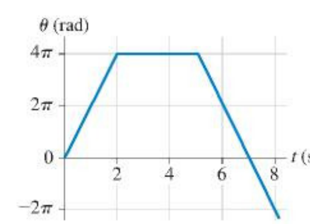


FIGURE EX4.24

24. | FIGURE EX4.24 shows the angular-position-versus-time graph for a particle moving in a circle. What is the particle's angular velocity at (a) $t = 1$ s, (b) $t = 4$ s, and (c) $t = 7$ s?

25. ‖ FIGURE EX4.25 shows the angular-velocity-versus-time graph for a particle moving in a circle, starting from $\theta_0 = 0$ rad at $t = 0$ s. Draw the angular-position-versus-time graph. Include an appropriate scale on both axes.

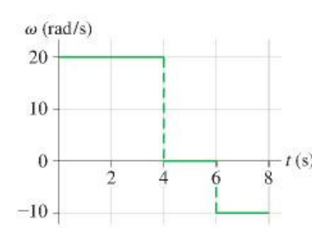


FIGURE EX4.25

26. ‖ The earth's radius is about 4000 miles. Kampala, the capital of Uganda, and Singapore are both nearly on the equator. The distance between them is 5000 miles. The flight from Kampala to Singapore takes 9.0 hours. What is the plane's angular velocity with respect to the earth's surface? Give your answer in °/h.

27. | An old-fashioned single-play vinyl record rotates on a turntable at 45 rpm. What are (a) the angular velocity in rad/s and (b) the period of the motion?

28. || As the earth rotates, what is the speed of (a) a physics student in Miami, Florida, at latitude 26°, and (b) a physics student in Fairbanks, Alaska, at latitude 65°? Ignore the revolution of the earth around the sun. The radius of the earth is 6400 km.

29. | How fast must a plane fly along the earth's equator so that the sun stands still relative to the passengers? In which direction must the plane fly, east to west or west to east? Give your answer in both km/h and mph. The earth's radius is 6400 km.

30. || A 3000-m-high mountain is located on the equator. How much faster does a climber on top of the mountain move than a surfer at a nearby beach? The earth's radius is 6400 km.

Section 4.5 Centripetal Acceleration

31. | Peregrine falcons are known for their maneuvering ability. In a BIO tight circular turn, a falcon can attain a centripetal acceleration 1.5 times the free-fall acceleration. What is the radius of the turn if the falcon is flying at 25 m/s?

32. | To withstand "g-forces" of up to 10 g's, caused by suddenly BIO pulling out of a steep dive, fighter jet pilots train on a "human centrifuge." 10 g's is an acceleration of 98 m/s^2. If the length of the centrifuge arm is 12 m, at what speed is the rider moving when she experiences 10 g's?

33. || The radius of the earth's very nearly circular orbit around the sun is 1.5×10^{11} m. Find the magnitude of the earth's (a) velocity, (b) angular velocity, and (c) centripetal acceleration as it travels around the sun. Assume a year of 365 days.

34. || A speck of dust on a spinning DVD has a centripetal acceleration of 20 m/s^2.
   a. What is the acceleration of a different speck of dust that is twice as far from the center of the disk?
   b. What would be the acceleration of the first speck of dust if the disk's angular velocity was doubled?

35. || Your roommate is working on his bicycle and has the bike upside down. He spins the 60-cm-diameter wheel, and you notice that a pebble stuck in the tread goes by three times every second. What are the pebble's speed and acceleration?

Section 4.6 Nonuniform Circular Motion

36. | FIGURE EX4.36 shows the angular velocity graph of the crankshaft in a car. What is the crankshaft's angular acceleration at (a) $t = 1$ s, (b) $t = 3$ s, and (c) $t = 5$ s?

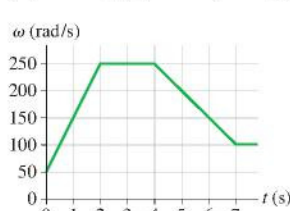


FIGURE EX4.36

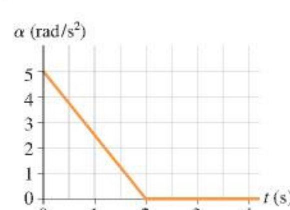


FIGURE EX4.37

37. || FIGURE EX4.37 shows the angular acceleration graph of a turntable that starts from rest. What is the turntable's angular velocity at (a) $t = 1$ s, (b) $t = 2$ s, and (c) $t = 3$ s?

38. || FIGURE EX4.38 shows the angular-velocity-versus-time graph for a particle moving in a circle. How many revolutions does the object make during the first 4 s?

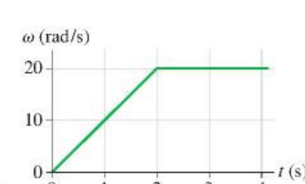


FIGURE EX4.38

39. || A wheel initially rotating at 60 rpm experiences the angular acceleration shown in FIGURE EX4.39. What is the wheel's angular velocity, in rpm, at $t = 3.0$ s?

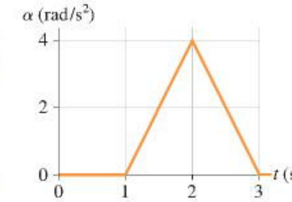


FIGURE EX4.39

40. || A 5.0-m-diameter merry-go-round is initially turning with a 4.0 s period. It slows down and stops in 20 s.
   a. Before slowing, what is the speed of a child on the rim?
   b. How many revolutions does the merry-go-round make as it stops?

41. || An electric fan goes from rest to 1800 rpm in 4.0 s. What is its angular acceleration?

42. || A bicycle wheel is rotating at 50 rpm when the cyclist begins to pedal harder, giving the wheel a constant angular acceleration of 0.50 rad/s^2.
   a. What is the wheel's angular velocity, in rpm, 10 s later?
   b. How many revolutions does the wheel make during this time?

43. || Starting from rest, a DVD steadily accelerates to 500 rpm in 1.0 s, rotates at this angular speed for 3.0 s, then steadily decelerates to a halt in 2.0 s. How many revolutions does it make?

Problems

44. ||| A spaceship maneuvering near Planet Zeta is located at $\vec{r} = (600\hat{\imath} - 400\hat{\jmath} + 200\hat{k}) \times 10^3$ km, relative to the planet, and traveling at $\vec{v} = 9500\hat{\imath}$ m/s. It turns on its thruster engine and accelerates with $\vec{a} = (40\hat{\imath} - 20\hat{k})$ m/s^2 for 35 min. What is the spaceship's position when the engine shuts off? Give your answer as a position vector measured in km.

45. ||| A particle moving in the xy-plane has velocity $\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}$ CALC at $t = 0$. It undergoes acceleration $\vec{a} = bt\hat{\imath} - cv_y\hat{\jmath}$, where b and c are constants. Find an expression for the particle's velocity at a later time t.

46. || A projectile's horizontal range over level ground is $v_0^2 \sin 2\theta/g$. At what launch angle or angles will the projectile land at half of its maximum possible range?

47. || a. A projectile is launched with speed v_0 and angle θ. Derive an expression for the projectile's maximum height h.
   b. A baseball is hit with a speed of 33.6 m/s. Calculate its height and the distance traveled if it is hit at angles of 30.0°, 45.0°, and 60.0°.

48. ||| A projectile is launched from ground level at angle θ and speed CALC v_0 into a headwind that causes a constant horizontal acceleration of magnitude a opposite the direction of motion.
   a. Find an expression in terms of a and g for the launch angle that gives maximum range.
   b. What is the angle for maximum range if a is 10% of g?

49. ||| A gray kangaroo can bound across level ground with each jump BIO carrying it 10 m from the takeoff point. Typically the kangaroo leaves the ground at a 20° angle. If this is so:
   a. What is its takeoff speed?
   b. What is its maximum height above the ground?

50. || A ball is thrown toward a cliff of height h with a speed of 30 m/s and an angle of 60° above horizontal. It lands on the edge of the cliff 4.0 s later.
   a. How high is the cliff?
   b. What was the maximum height of the ball?
   c. What is the ball's impact speed?

51. || A tennis player hits a ball 2.0 m above the ground. The ball leaves his racquet with a speed of 20.0 m/s at an angle 5.0° above the horizontal. The horizontal distance to the net is 7.0 m, and the net is 1.0 m high. Does the ball clear the net? If so, by how much? If not, by how much does it miss?

52. || You are target shooting using a toy gun that fires a small ball at a speed of 15 m/s. When the gun is fired at an angle of 30° above horizontal, the ball hits the bull's-eye of a target at the same height as the gun. Then the target distance is halved. At what angle must you aim the gun to hit the bull's-eye in its new position? (Mathematically there are two solutions to this problem; the physically reasonable answer is the smaller of the two.)

53. || A 35 g steel ball is held by a ceiling-mounted electromagnet 3.5 m above the floor. A compressed-air cannon sits on the floor, 4.0 m to one side of the point directly under the ball. When a button is pressed, the ball drops and, simultaneously, the cannon fires a 25 g plastic ball. The two balls collide 1.0 m above the floor. What was the launch speed of the plastic ball?

54. || You are watching an archery tournament when you start wondering how fast an arrow is shot from the bow. Remembering your physics, you ask one of the archers to shoot an arrow parallel to the ground. You find the arrow stuck in the ground 60 m away, making a 3.0° angle with the ground. How fast was the arrow shot?

55. || You're 6.0 m from one wall of the house seen in FIGURE P4.55. You want to toss a ball to your friend who is 6.0 m from the opposite wall. The throw and catch each occur 1.0 m above the ground.
  a. What minimum speed will allow the ball to clear the roof?
  b. At what angle should you toss the ball?

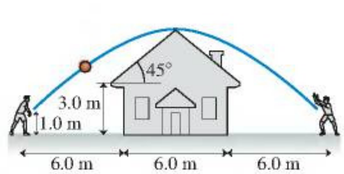


FIGURE P4.55

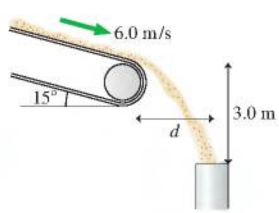


FIGURE P4.56

56. || Sand moves without slipping at 6.0 m/s down a conveyer that is tilted at 15°. The sand enters a pipe 3.0 m below the end of the conveyer belt, as shown in FIGURE P4.56. What is the horizontal distance d between the conveyer belt and the pipe?

57. || A stunt man drives a car at a speed of 20 m/s off a 30-m-high cliff. The road leading to the cliff is inclined upward at an angle of 20°.
  a. How far from the base of the cliff does the car land?
  b. What is the car's impact speed?

58. ||| BIO A javelin thrower standing at rest holds the center of the javelin behind her head, then accelerates it through a distance of 70 cm as she throws. She releases the javelin 2.0 m above the ground traveling at an angle of 30° above the horizontal. Top-rated javelin throwers do throw at about a 30° angle, not the 45° you might have expected, because the biomechanics of the arm allow them to throw the javelin much faster at 30° than they would be able to at 45°. In this throw, the javelin hits the ground 62 m away. What was the acceleration of the javelin during the throw? Assume that it has a constant acceleration.

59. || A rubber ball is dropped onto a ramp that is tilted at 20°, as shown in FIGURE P4.59. A bouncing ball obeys the "law of reflection," which says that the ball leaves the surface at the same angle it approached the surface. The ball's next bounce is 3.0 m to the right of its first bounce. What is the ball's rebound speed on its first bounce?

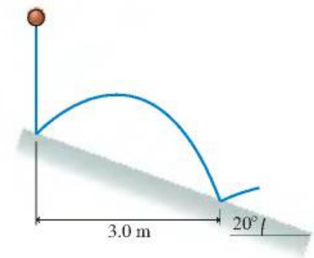


FIGURE P4.59

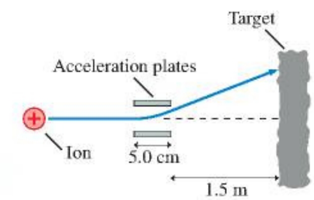


FIGURE P4.60

60. ||| BIO You are asked to consult for the city's research hospital, where a group of doctors is investigating the bombardment of cancer tumors with high-energy ions. As FIGURE P4.60 shows, ions are fired directly toward the center of the tumor at speeds of 5.0×10^6 m/s. To cover the entire tumor area, the ions are deflected sideways by passing them between two charged metal plates that accelerate the ions perpendicular to the direction of their initial motion. The acceleration region is 5.0 cm long, and the ends of the acceleration plates are 1.5 m from the target. What sideways acceleration is required to deflect an ion 2.0 cm to one side?

61. || Ships A and B leave port together. For the next two hours, ship A travels at 20 mph in a direction 30° west of north while ship B travels 20° east of north at 25 mph.
  a. What is the distance between the two ships two hours after they depart?
  b. What is the speed of ship A as seen by ship B?

62. || While driving north at 25 m/s during a rainstorm you notice that the rain makes an angle of 38° with the vertical. While driving back home moments later at the same speed but in the opposite direction, you see that the rain is falling straight down. From these observations, determine the speed and angle of the raindrops relative to the ground.

63. || You've been assigned the task of using a shaft encoder—a device that measures the angle of a shaft or axle and provides a signal to a computer—to analyze the rotation of an engine crankshaft under certain conditions. The table lists the crankshaft's angles over a 0.6 s interval.

Time (s)	Angle (rad)
0.0	0.0
0.1	2.0
0.2	3.2
0.3	4.3
0.4	5.3
0.5	6.1
0.6	7.0

Is the crankshaft rotating with uniform circular motion? If so, what is its angular velocity in rpm? If not, is the angular acceleration positive or negative?

64. || A circular track has several concentric rings where people can run at their leisure. Phil runs on the outermost track with radius r_P while Annie runs on an inner track with radius $r_A = 0.80r_P$. The runners start side by side, along a radial line, and run at the same speed in a counterclockwise direction. How many revolutions has Annie made when Annie's and Phil's velocity vectors point in opposite directions for the first time?

65. ||| A typical laboratory centrifuge rotates at 4000 rpm. Test tubes have to be placed into a centrifuge very carefully because of the very large accelerations.
   a. What is the acceleration at the end of a test tube that is 10 cm from the axis of rotation?
   b. For comparison, what is the magnitude of the acceleration a test tube would experience if dropped from a height of 1.0 m and stopped in a 1.0-ms-long encounter with a hard floor?

66. || Astronauts use a centrifuge to simulate the acceleration of a
BIO rocket launch. The centrifuge takes 30 s to speed up from rest to its top speed of 1 rotation every 1.3 s. The astronaut is strapped into a seat 6.0 m from the axis.
   a. What is the astronaut's tangential acceleration during the first 30 s?
   b. How many g's of acceleration does the astronaut experience when the device is rotating at top speed? Each 9.8 m/s^2 of acceleration is 1 g.

67. || Communications satellites are placed in a circular orbit where they stay directly over a fixed point on the equator as the earth rotates. These are called *geosynchronous orbits*. The radius of the earth is 6.37×10^6 m, and the altitude of a geosynchronous orbit is 3.58×10^7 m ($\approx$22,000 miles). What are (a) the speed and (b) the magnitude of the acceleration of a satellite in a geosynchronous orbit?

68. || A computer hard disk 8.0 cm in diameter is initially at rest. A small dot is painted on the edge of the disk. The disk accelerates at 600 rad/s^2 for $\frac{1}{2}$ s, then coasts at a steady angular velocity for another $\frac{1}{2}$ s.
   a. What is the speed of the dot at $t = 1.0$ s?
   b. Through how many revolutions has the disk turned?

69. || A high-speed drill rotating ccw at 2400 rpm comes to a halt in 2.5 s.
   a. What is the magnitude of the drill's angular acceleration?
   b. How many revolutions does it make as it stops?

70. || A turbine is spinning at 3800 rpm. Friction in the bearings is so low that it takes 10 min to coast to a stop. How many revolutions does the turbine make while stopping?

71. || Your 64-cm-diameter car tire is rotating at 3.5 rev/s when suddenly you press down hard on the accelerator. After traveling 200 m, the tire's rotation has increased to 6.0 rev/s. What was the tire's angular acceleration? Give your answer in rad/s^2.

72. || The angular velocity of a process control motor is
CALC $\omega = \left(20 - \frac{1}{2}t^2\right)$ rad/s, where t is in seconds.
   a. At what time does the motor reverse direction?
   b. Through what angle does the motor turn between $t = 0$ s and the instant at which it reverses direction?

73. || A Ferris wheel of radius R speeds up with angular acceleration α starting from rest. Find an expression for the (a) velocity and (b) centripetal acceleration of a rider after the Ferris wheel has rotated through angle $\Delta\theta$.

74. || A 6.0-cm-diameter gear rotates with angular velocity $\omega =$
CALC $\left(2.0 + \frac{1}{2}t^2\right)$ rad/s, where t is in seconds. At $t = 4.0$ s, what are:
   a. The gear's angular acceleration?
   b. The tangential acceleration of a tooth on the gear?

75. || A painted tooth on a spinning gear has angular acceleration
CALC $\alpha = (20 - t)$ rad/s^2, where t is in s. Its initial angular velocity, at $t = 0$ s, is 300 rpm. What is the tooth's angular velocity in rpm at $t = 20$ s?

76. ||| A car starts from rest on a curve with a radius of 120 m and accelerates tangentially at 1.0 m/s^2. Through what angle will the car have traveled when the magnitude of its total acceleration is 2.0 m/s^2?

77. ||| A long string is wrapped around a 6.0-cm-diameter cylinder, initially at rest, that is free to rotate on an axle. The string is then pulled with a constant acceleration of 1.5 m/s^2 until 1.0 m of string has been unwound. If the string unwinds without slipping, what is the cylinder's angular speed, in rpm, at this time?

In Problems 78 through 80 you are given the equations that are used to solve a problem. For each of these, you are to
   a. Write a realistic problem for which these are the correct equations. Be sure that the answer your problem requests is consistent with the equations given.
   b. Finish the solution of the problem, including a pictorial representation.

78. $100 \text{ m} = 0 \text{ m} + (50\cos\theta \text{ m/s})t_1$
    $0 \text{ m} = 0 \text{ m} + (50\sin\theta \text{ m/s})t_1 - \frac{1}{2}(9.80 \text{ m/s}^2)t_1^{\,2}$

79. $v_x = -(6.0\cos 45°) \text{ m/s} + 3.0 \text{ m/s}$
    $v_y = (6.0 \sin 45°) \text{ m/s} + 0 \text{ m/s}$
    $100 \text{ m} = v_y t_1,\ x_1 = v_x t_1$

80. $2.5 \text{ rad} = 0 \text{ rad} + \omega_i(10 \text{ s}) + \left((1.5 \text{ m/s}^2)/2(50 \text{ m})\right)(10 \text{ s})^2$
    $\omega_f = \omega_i + \left((1.5 \text{ m/s}^2)/(50 \text{ m})\right)(10 \text{ s})$

Challenge Problems

81. ||| In one contest at the county fair, seen in FIGURE CP4.81, a spring-loaded plunger launches a ball at a speed of 3.0 m/s from one corner of a smooth, flat board that is tilted up at a 20° angle. To win, you must make the ball hit a small target at the adjacent corner, 2.50 m away. At what angle θ should you tilt the ball launcher?

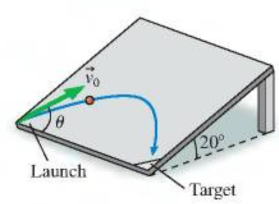


FIGURE CP4.81

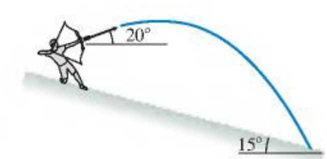


FIGURE CP4.82

82. ||| An archer standing on a 15° slope shoots an arrow 20° above the horizontal, as shown in FIGURE CP4.82. How far down the slope does the arrow hit if it is shot with a speed of 50 m/s from 1.75 m above the ground?

83. ||| A skateboarder starts up a 1.0-m-high, 30° ramp at a speed of 7.0 m/s. The skateboard wheels roll without friction. At the top she leaves the ramp and sails through the air. How far from the end of the ramp does the skateboarder touch down?

84. ||| A cannon on a train car fires a projectile to the right with speed
CALC v_0, relative to the train, from a barrel elevated at angle θ. The cannon fires just as the train, which had been cruising to the right along a level track with speed v_{train}, begins to accelerate with acceleration a, which can be either positive (speeding up) or negative (slowing down). Find an expression for the angle at which the projectile should be fired so that it lands as far as possible from the cannon. You can ignore the small height of the cannon above the track.

85. ||| A child in danger of drowning in a river is being carried downstream by a current that flows uniformly with a speed of 2.0 m/s. The child is 200 m from the shore and 1500 m upstream of the boat dock from which the rescue team sets out. If their boat speed is 8.0 m/s with respect to the water, at what angle from the shore should the pilot leave the shore to go directly to the child?